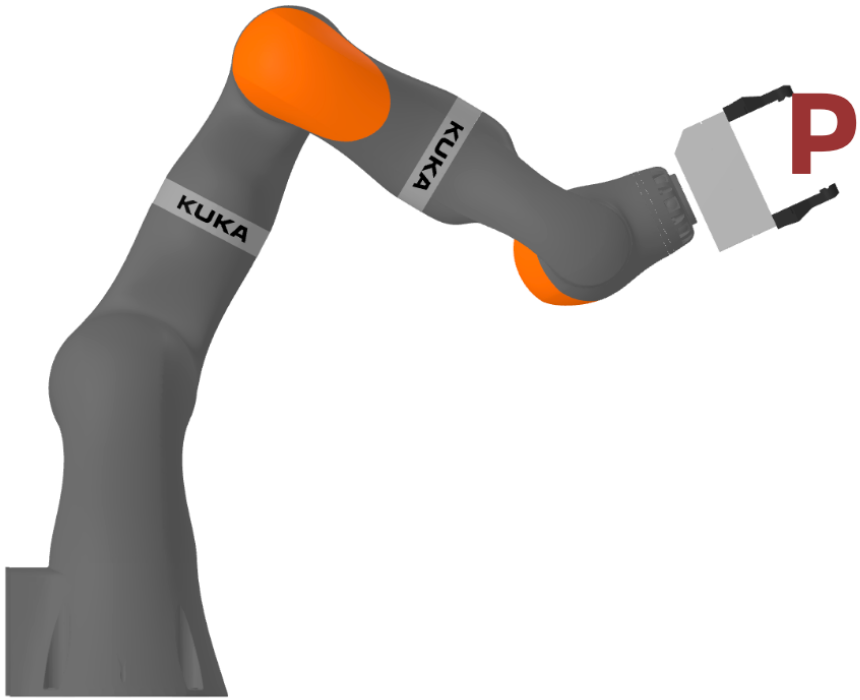


Position Control

MIT 6.4210/2

Robotic Manipulation

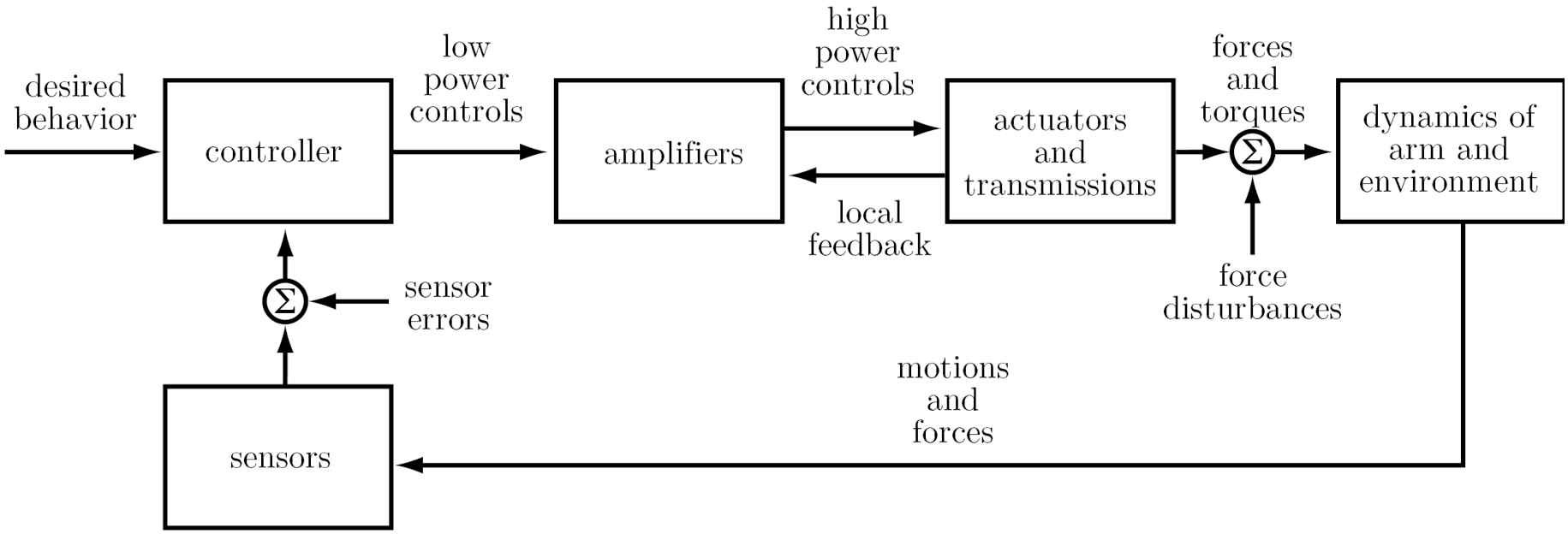
Fall 2025, Lecture 12

Massachusetts Institute of Technology

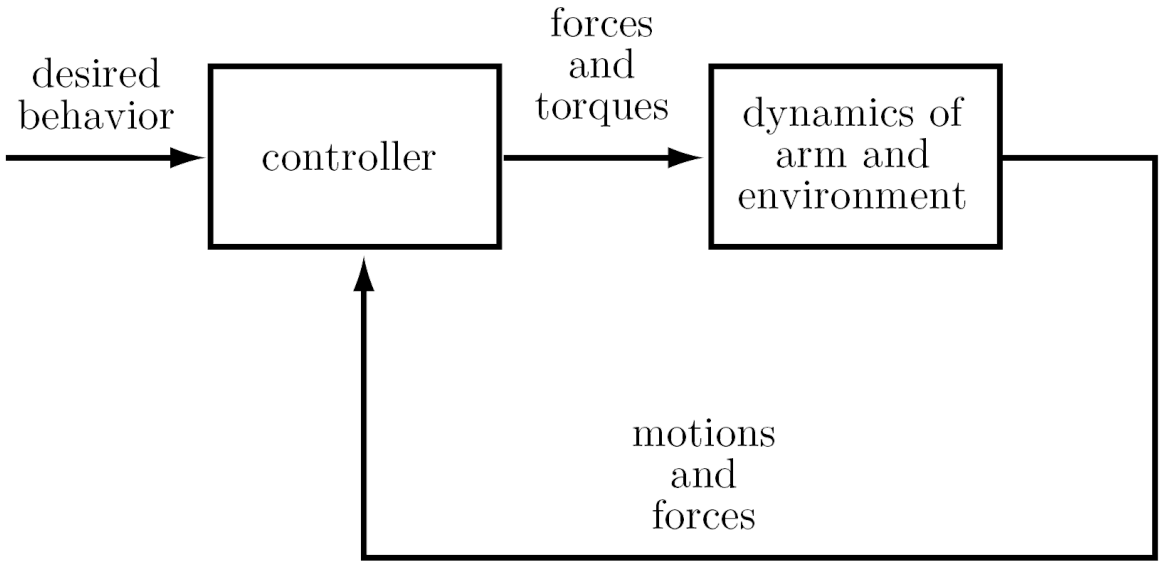
Example control objectives:

- **Motion (position/velocity) control**
- **force control**
- **hybrid motion-force control**
- **impedance control**

Control system block diagram:



Simplified block diagram:



Also assuming continuous-time (not discrete-time) control.

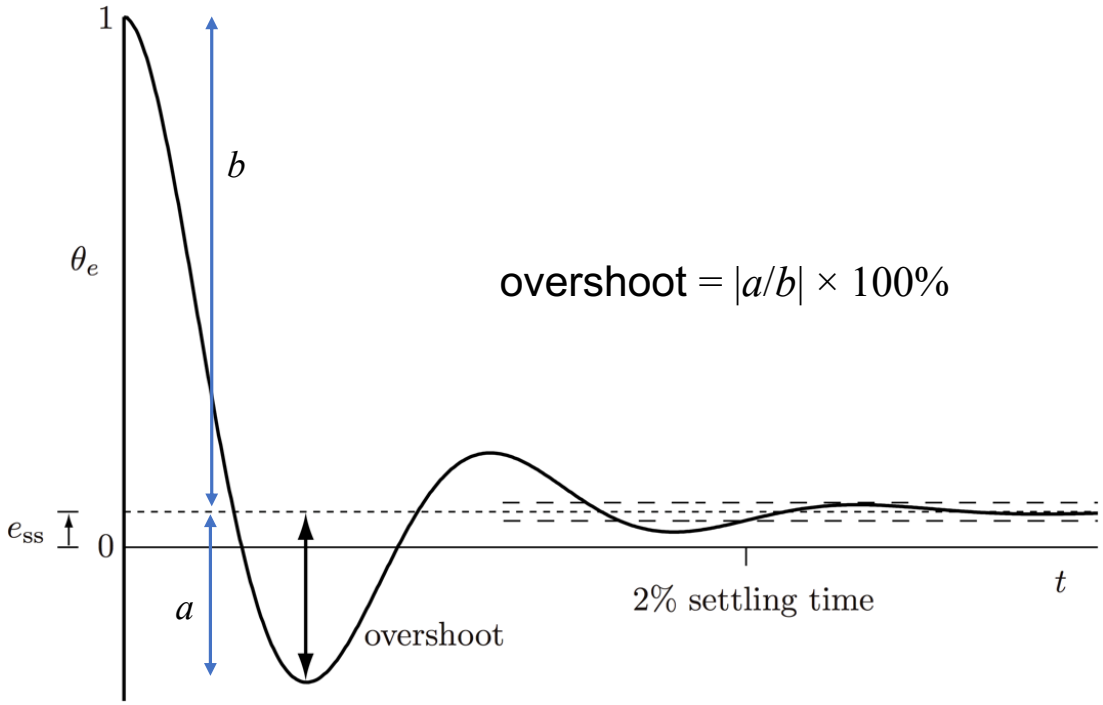
For motion control,

reference: $\theta_d(t)$
actual: $\theta(t)$
error: $\theta_e(t) = \theta_d(t) - \theta(t)$

Unit step error response:
$\theta_e(t)$ starting from $\theta_e(0) = 1$

Steady-state error response: e_{ss}

Transient error response:
overshoot, settling time



overshoot = $|a/b| \times 100\%$

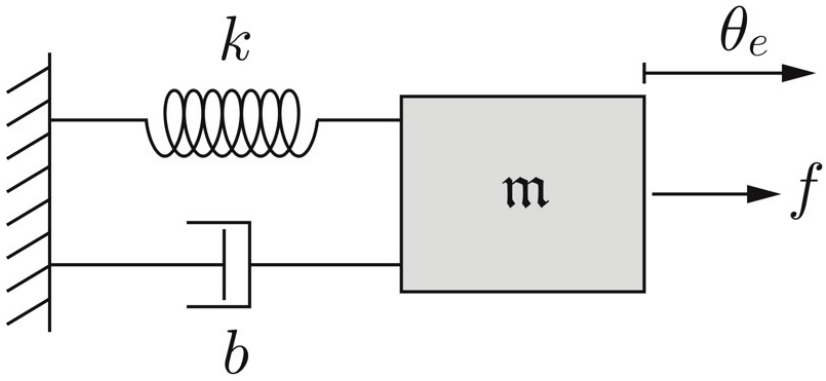
System dynamics, feedback controllers, and error response are often modeled by **linear ordinary differential equations**.

The simplest linear ODE exhibiting overshoot is second order, e.g.,

$$\mathfrak{m}\ddot{\theta}_e + b\dot{\theta}_e + k\theta_e = f$$

or, if $f = 0$,

$$\ddot{\theta}_e + \frac{b}{\mathfrak{m}}\dot{\theta}_e + \frac{k}{\mathfrak{m}}\theta_e = 0$$



k and b depend on the control law

A more general p^{th}-order linear ODE:

$$a_p\theta_e^{(p)} + a_{p-1}\theta_e^{(p-1)} + \cdots + a_2\ddot{\theta}_e + a_1\dot{\theta}_e + a_0\theta_e = c \quad \textbf{nonhomogenous}$$

$$a_p\theta_e^{(p)} + a_{p-1}\theta_e^{(p-1)} + \cdots + a_2\ddot{\theta}_e + a_1\dot{\theta}_e + a_0\theta_e = 0 \quad \textbf{homogeneous}$$

$$\theta_e^{(p)} + a'_{p-1}\theta_e^{(p-1)} + \cdots + a'_2\ddot{\theta}_e + a'_1\dot{\theta}_e + a'_0\theta_e = 0$$

$$\theta_e^{(p)} = -a'_{p-1}\theta_e^{(p-1)} - \cdots - a'_2\ddot{\theta}_e - a'_1\dot{\theta}_e - a'_0\theta_e$$

Defining a state vector $x = (x_1, x_2, ..., x_p)$, you can write the p^{th}-order ODE as p first-order ODEs (a vector ODE).

$$
\begin{aligned}
x_1 &= \theta_e, \\
x_2 &= \dot{x}_1 = \dot{\theta}_e, \\
&\vdots \quad \vdots \\
x_p &= \dot{x}_{p-1} = \theta_e^{(p-1)}
\end{aligned}
$$

$$\dot{x}(t) = Ax(t) \rightarrow x(t) = e^{At}x(0)$$

$$
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-a_0' & -a_1' & -a_2' & \cdots & -a_{p-2}' & -a_{p-1}'
\end{bmatrix} \in \mathbb{R}^{p \times p}
$$

$$\dot{x}(t) = Ax(t) \rightarrow x(t) = e^{At}x(0)$$

If $\mathrm{Re}(s) < 0$ for all eigenvalues s of A, then the error dynamics are **stable** (the error decays to zero).

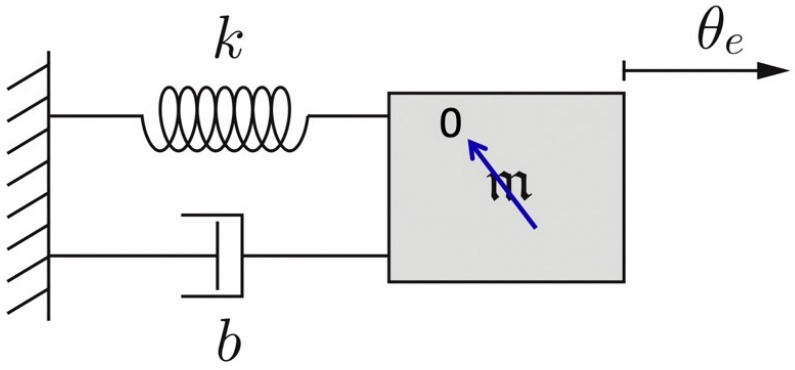
The eigenvalues are the roots of the **characteristic equation**

$$\det(sI - A) = s^p + a'_{p-1}s^{p-1} + \cdots + a'_2 s^2 + a'_1 s + a'_0 = 0$$

Necessary conditions for stability: each $a'_i > 0$.

These necessary conditions are also **sufficient** for first- and second-order systems.

First-order error dynamics



0

$$m\ddot{\theta}_e + b\dot{\theta}_e + k\theta_e = 0$$

$$\dot{\theta}_e(t) + \frac{k}{b}\theta_e(t) = 0$$

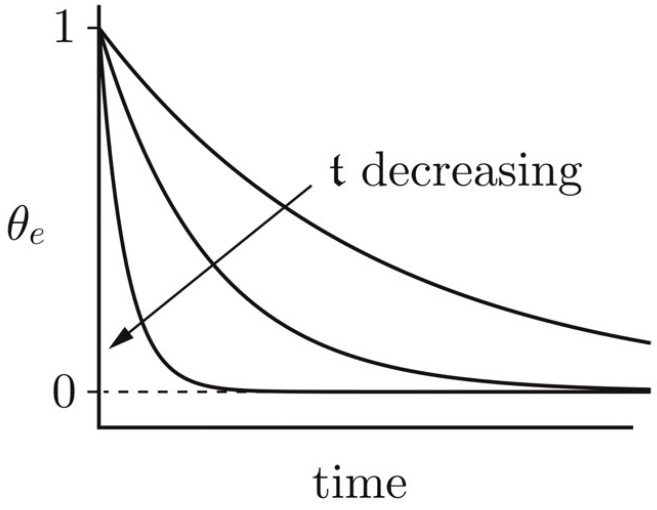
standard first-order form

time constant

$$t = b/k$$

$$\dot{\theta}_e(t) + \frac{1}{t}\theta_e(t) = 0$$

First-order error dynamics



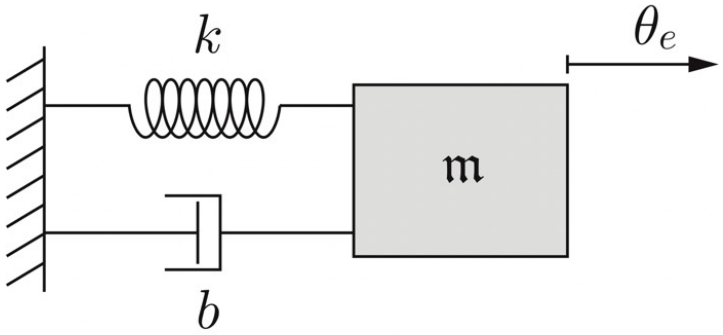
$$\theta_e(0) = 1$$

$$\theta_e(t) = e^{-t/\mathfrak{t}}\theta_e(0)$$

$$\frac{\theta_e(t)}{\theta_e(0)} = 0.02 = e^{-t/\mathfrak{t}}$$

$$\ln 0.02 = -t/\mathfrak{t} \quad \rightarrow \quad t = 3.91\mathfrak{t}$$

Second-order error dynamics



$$\ddot{\theta}_e(t) + \frac{b}{\mathfrak{m}}\dot{\theta}_e(t) + \frac{k}{\mathfrak{m}}\theta_e(t) = 0$$

natural frequency **damping ratio**

$$\omega_n = \sqrt{k/\mathfrak{m}} \qquad \zeta = b/(2\sqrt{k\mathfrak{m}})$$

$$\ddot{\theta}_e(t) + 2\zeta\omega_n\dot{\theta}_e(t) + \omega_n^2\theta_e(t) = 0$$

standard second-order form

Second-order error dynamics

$$\ddot{\theta}_e(t) + 2\zeta\omega_n\dot{\theta}_e(t) + \omega_n^2\theta_e(t) = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$\zeta > 1$: **Overdamped**

$\zeta = 1$: **Critically damped**

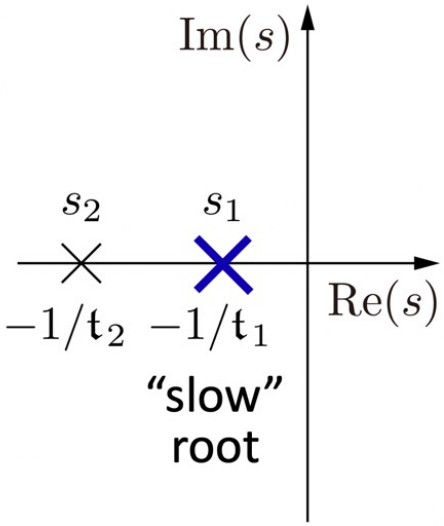
$\zeta < 1$: **Underdamped**

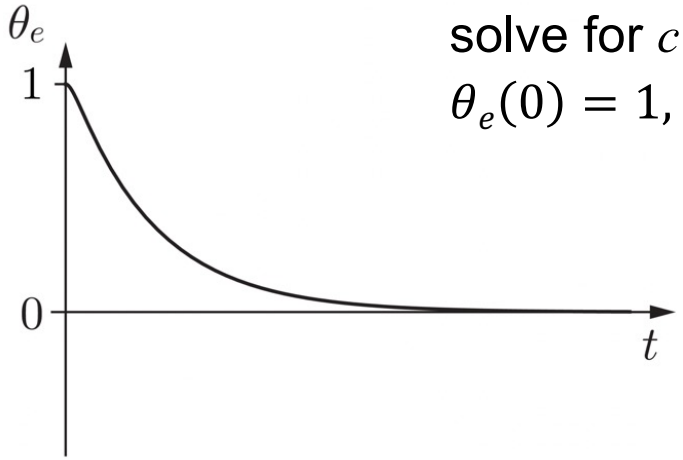
$\zeta > 1$: Overdamped

$$\theta_e(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$$

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

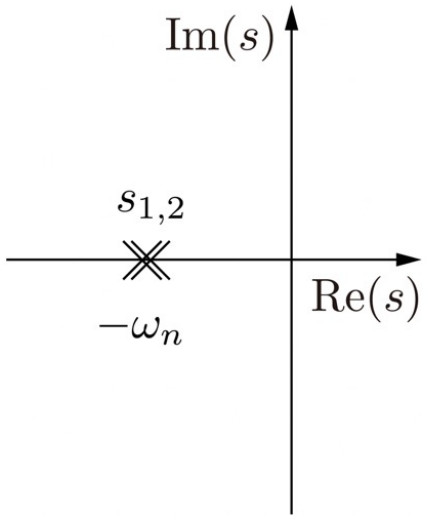


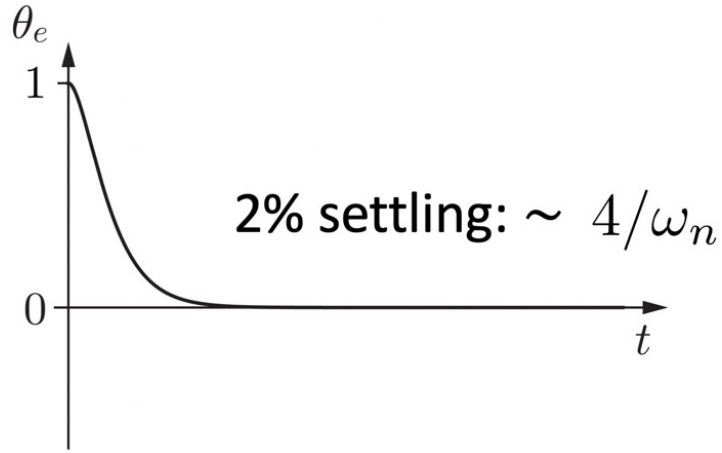


solve for c_1 and c_2 using
$\theta_e(0) = 1, \dot{\theta}_e(0) = 0$

$\zeta = 1$: Critically damped

$$\theta_e(t) = (c_1 + c_2 t)e^{-\omega_n t} \qquad s_{1,2} = -\omega_n$$





2% settling: ~ $4/\omega_n$

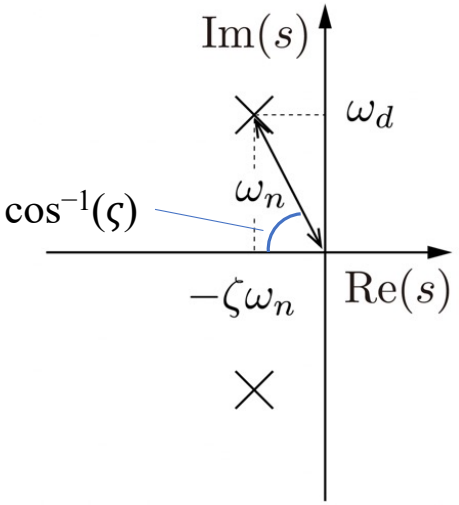
$\zeta < 1$: Underdamped

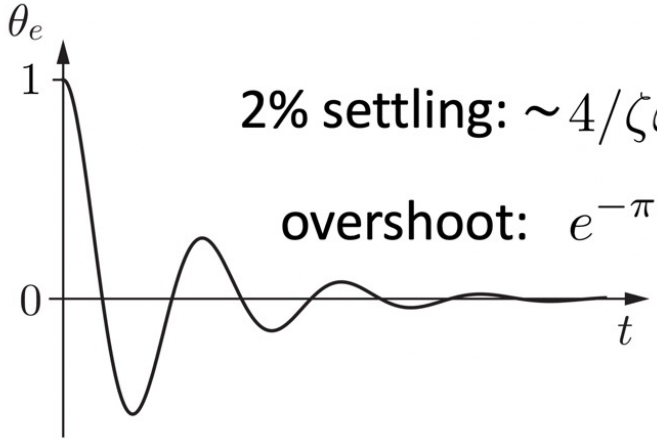
$$\theta_e(t) = (c_1 \cos \omega_d t + c_2 \sin \omega_d t)\, e^{-\zeta \omega_n t}$$

damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

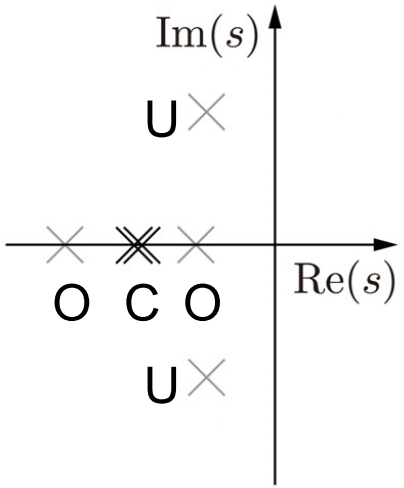
$$s_{1,2} = -\zeta \omega_n \pm j \omega_d$$

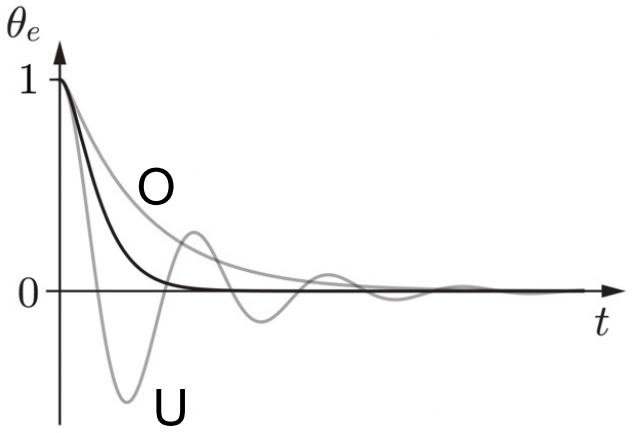




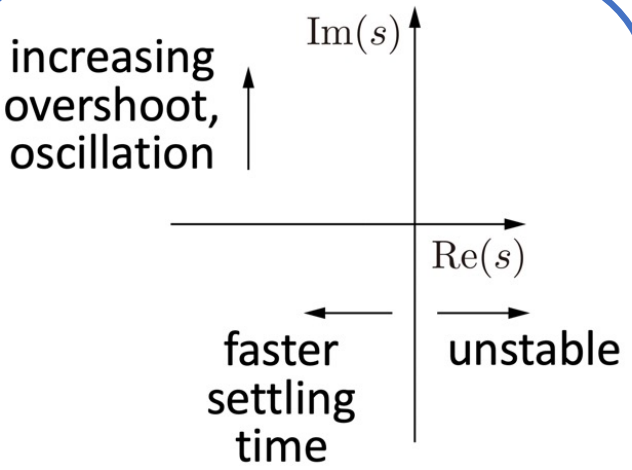
2% settling: $\sim 4/\zeta\omega_n$

overshoot: $e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$





second-order systems



time response relative to root locations for general systems

For a single joint with the joint velocity as the control:

- **Open-loop (feedforward) control**:

$$\dot{\theta}(t) = \dot{\theta}_d(t)$$

- **Closed-loop (feedback) control**:

$$\dot{\theta}(t) = f(\theta_d(t), \theta(t))$$

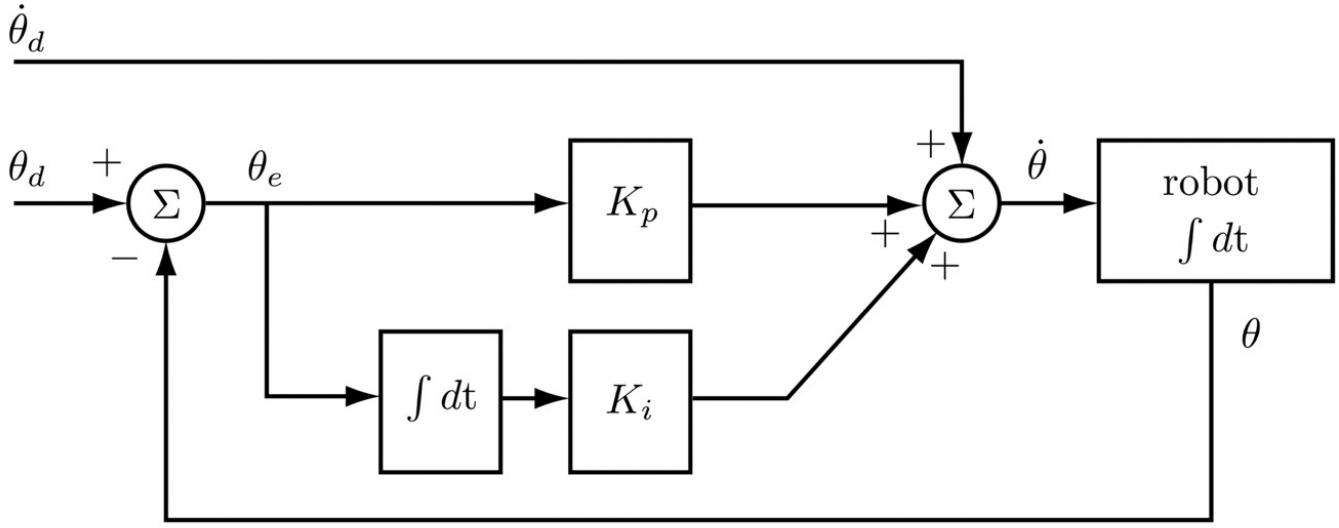
- **FF + Proportional-Integral (PI) FB control**:

$$\dot{\theta}(t) = \dot{\theta}_d(t) + K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, d\mathrm{t}\ ,\ K_p, K_i \geq 0$$

What is the point of FF control in this control law?

- *reduces to FF control if* $K_p, K_i = 0$
- *if no FF term:* **P control** *when* $K_i = 0$, **I control** *when* $K_p = 0$

Block diagram



$$\dot{\theta}(t) = \dot{\theta}_d(t) + K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, d\mathrm{t}$$

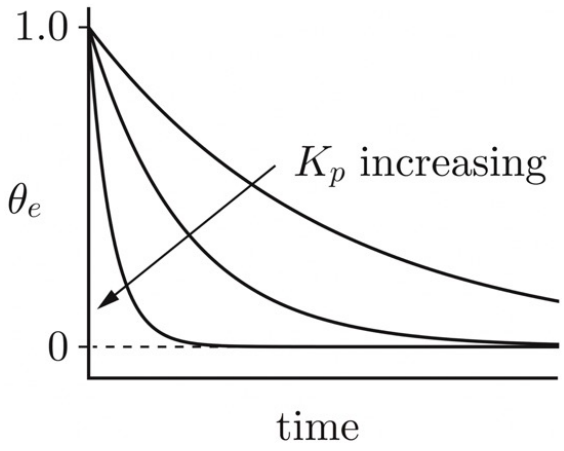
Setpoint control, $\theta_d(t) = c$, with a P controller

$$\dot{\theta}_e(t) = \cancelto{0}{\dot{\theta}_d(t)} - \dot{\theta}(t)$$

$$\dot{\theta}_e(t) = -K_p\theta_e(t) \quad \rightarrow \quad \dot{\theta}_e(t) + K_p\theta_e(t) = 0$$

$$\mathrm{t} = 1/K_p$$

$$\theta_e(t) = e^{-t/\mathrm{t}}\theta_e(0)$$



Constant velocity control, $\theta_d(t) = ct + a$

P control

$$\dot{\theta}_e(t) = \dot{\theta}_d(t) - \dot{\theta}(t) = c - K_p\theta_e(t)$$

$$\dot{\theta}_e(t) + K_p\theta_e(t) = c$$

$$\theta_e(t) = \frac{c}{K_p} + \left(\theta_e(0) - \frac{c}{K_p}\right) e^{-K_p t}$$

PI control

$$\dot{\theta}(t) = K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, d\mathrm{t}$$

$$\dot{\theta}_e(t) + K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\, d\mathrm{t} = c$$

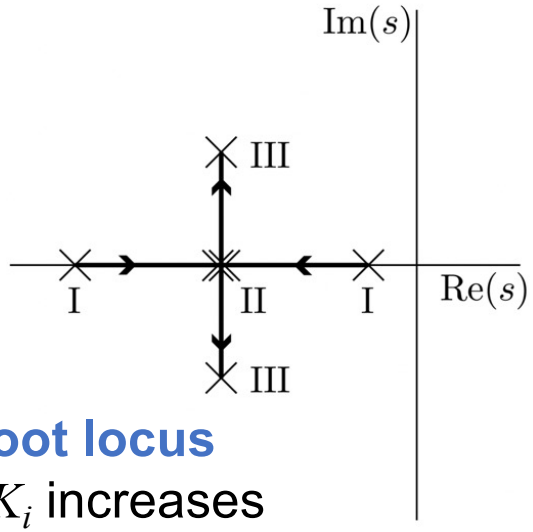
$$\ddot{\theta}_e(t) + K_p\dot{\theta}_e(t) + K_i\theta_e(t) = 0$$

$$\zeta = K_p/(2\sqrt{K_i})$$

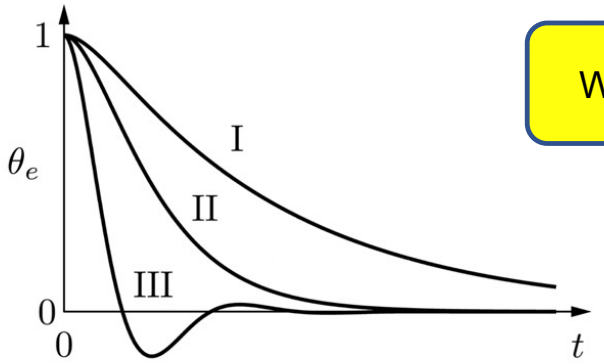
$$\omega_n = \sqrt{K_i}$$

characteristic equation of PI error dynamics: $s^2 + K_p s + K_i = 0$

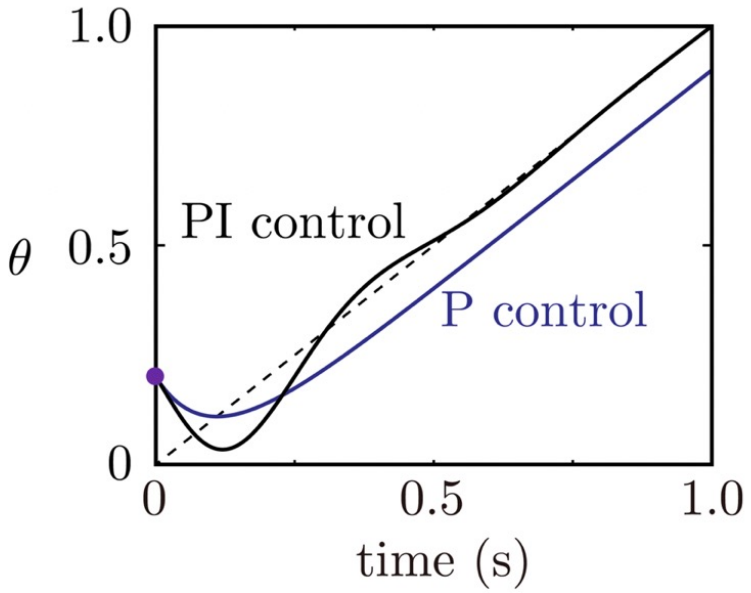
$$s_{1,2} = -\frac{K_p}{2} \pm \sqrt{\frac{K_p^2}{4} - K_i}$$

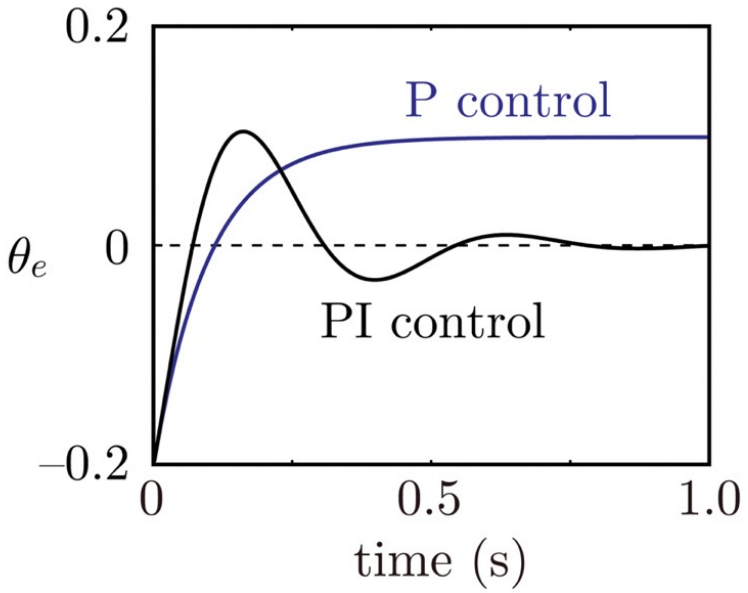


root locus
as K_i increases

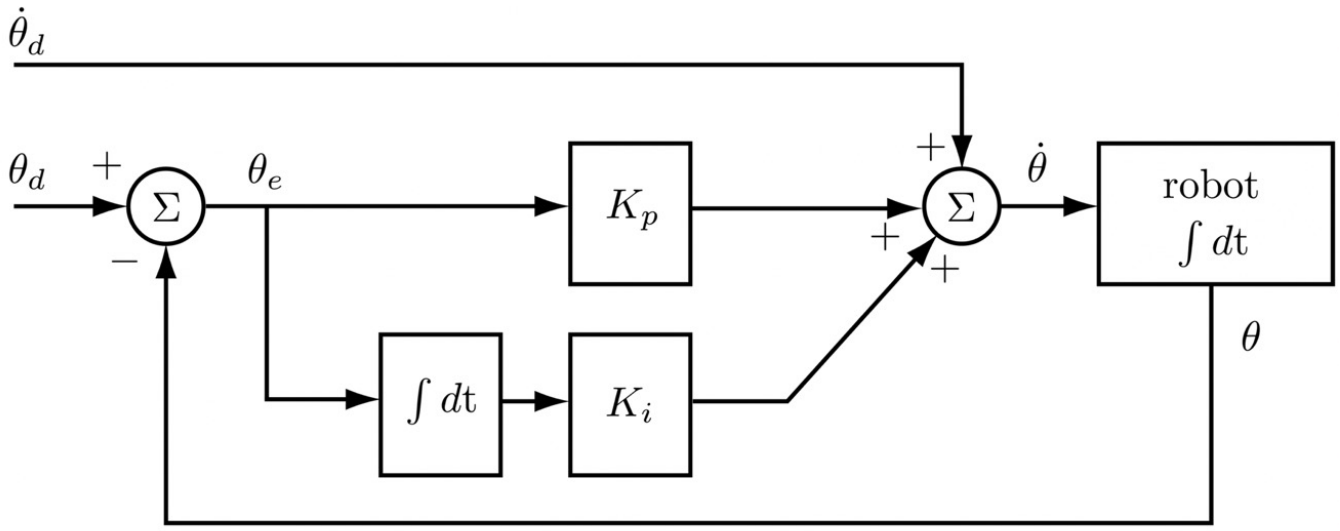


What if $K_i = 0$?

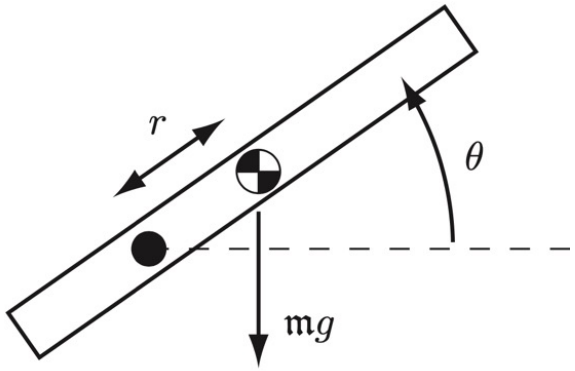

1.0
0.5
0
θ
PI control
P control
0
0.5
1.0
time (s)


0.2
0
−0.2
θ_e
P control
PI control
0
0.5
1.0
time (s)

Multi-joint control: θ, θ_d, and θ_e are vectors and $K_p = k_p I,\ K_i = k_i I$



$$\dot{\theta}(t) = \dot{\theta}_d(t) + K_p\theta_e(t) + K_i \int_0^t \theta_e(\mathrm{t})\ d\mathrm{t}$$

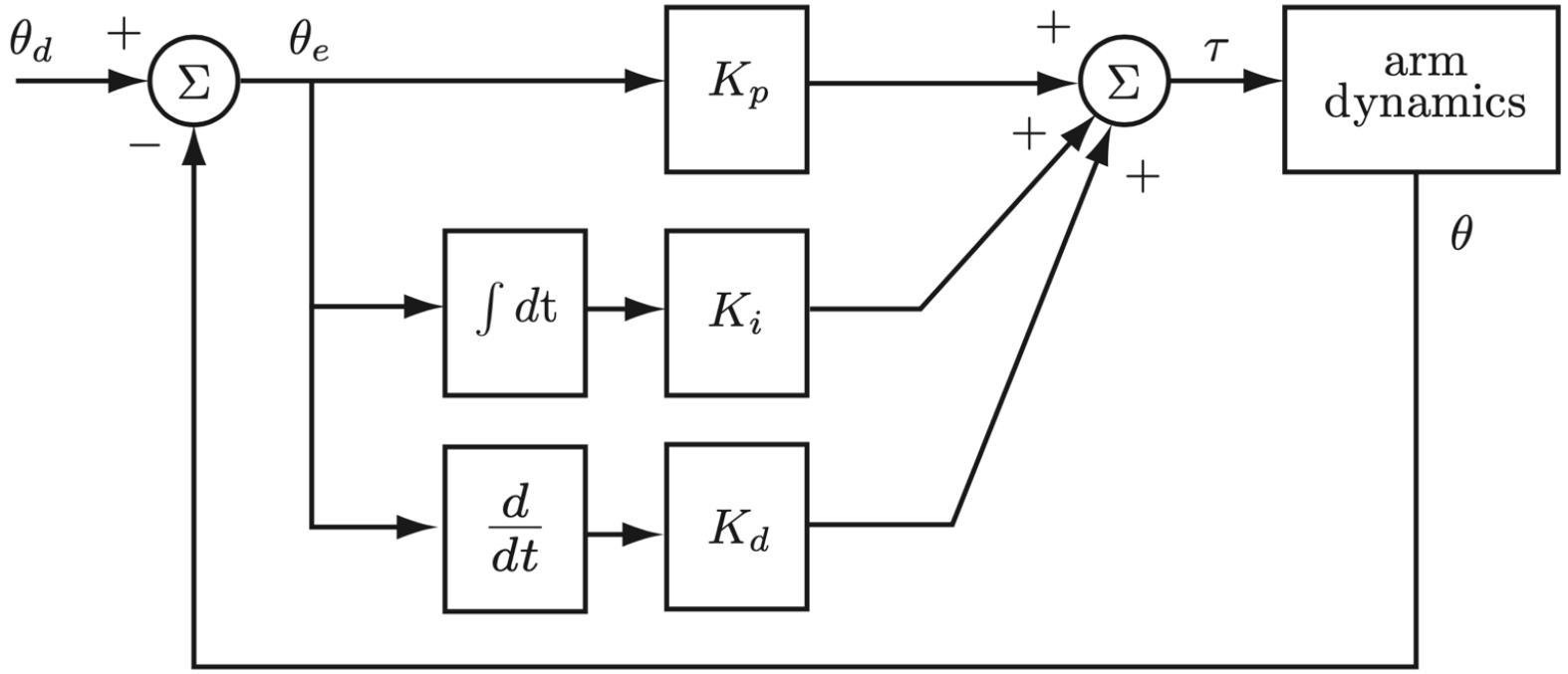


$$\tau = M\ddot{\theta} + \mathfrak{m}gr\cos\theta + b\dot{\theta}$$

$$\tau = M\ddot{\theta} + h(\theta, \dot{\theta})$$

Proportional-Integral-Derivative (PID) control

$$\tau = K_p\theta_e + K_i \int \theta_e(\mathrm{t})d\mathrm{t} + K_d\dot{\theta}_e$$


θ_d
+
−
Σ
θ_e
K_p
$\int dt$
K_i
$\frac{d}{dt}$
K_d
+
+
+
Σ
τ
arm dynamics
θ

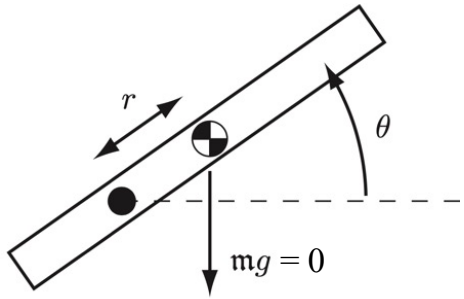
Setpoint PD control, $g = 0$

$$\tau = K_p\theta_e + \cancel{K_i}^{0} \int \theta_e(\mathrm{t})d\mathrm{t} + K_d\dot{\theta}_e$$

$$\tau = M\ddot{\theta} + \cancel{\mathrm{m}g}^{0} r\cos\theta + b\dot{\theta}$$

$$\dot{\theta}_d = \ddot{\theta}_d = 0$$

$$M\ddot{\theta} + b\dot{\theta} = K_p\theta_e + K_d\dot{\theta}_e$$



$$\ddot{\theta}_e + \frac{b + K_d}{M}\dot{\theta}_e + \frac{K_p}{M}\theta_e = 0$$

$$\zeta = \frac{b + K_d}{2\sqrt{K_p M}} \qquad \omega_n = \sqrt{\frac{K_p}{M}}$$

For what gains are the error dynamics stable?

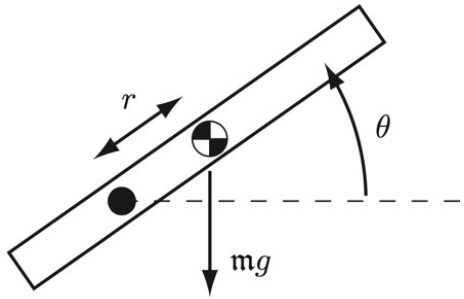
Setpoint PD control, $g \neq 0$

$$\tau = K_p\theta_e + \cancel{K_i \int \theta_e(\mathrm{t})d\mathrm{t}}^{0} + K_d\dot{\theta}_e$$

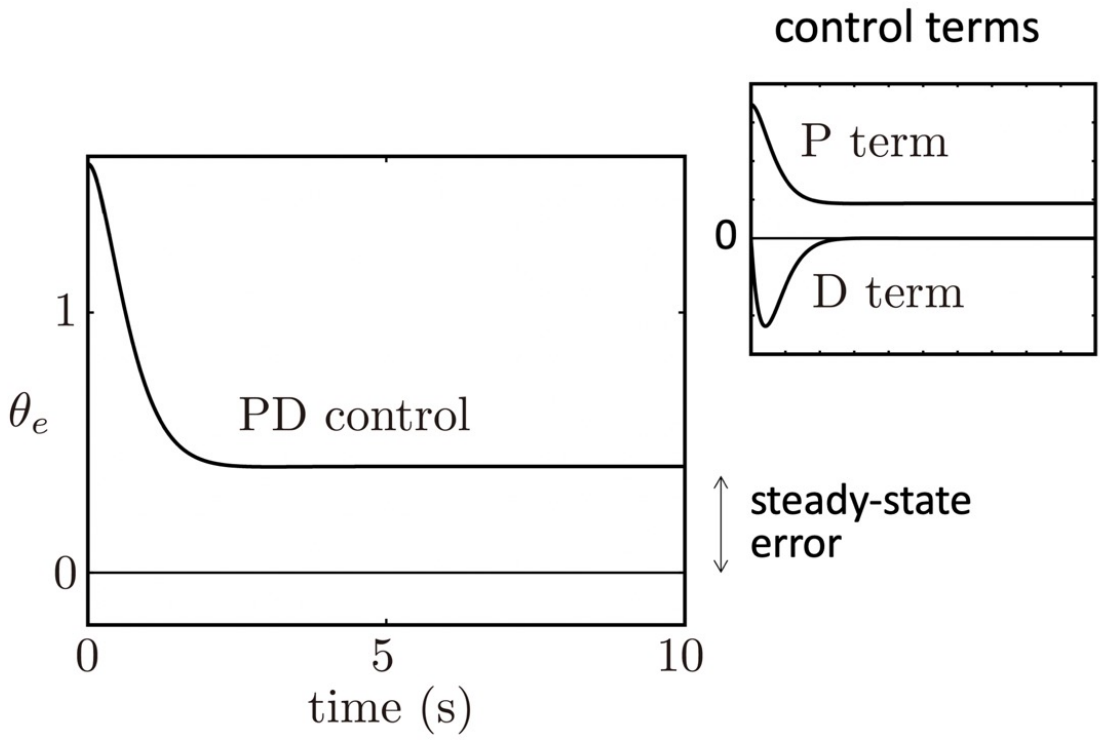
$$\tau = M\ddot{\theta} + \mathfrak{m}gr\cos\theta + b\dot{\theta}$$

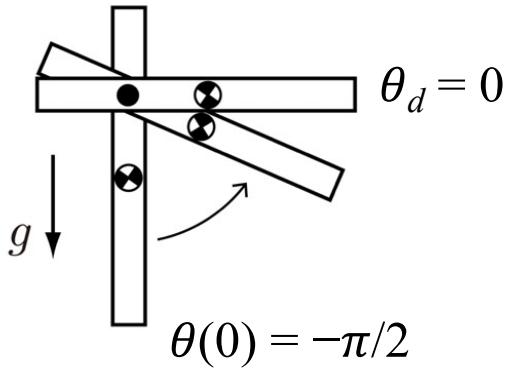
$$\dot{\theta}_d = \ddot{\theta}_d = 0$$

$$M\ddot{\theta}_e + (b + K_d)\dot{\theta}_e + K_p\theta_e = \mathfrak{m}gr\cos\theta$$



Nonhomogeneous. What is the steady-state error?


control terms
P term
0
D term
1
θ_e
PD control
steady-state
error
0
0
5
10
time (s)


$\theta_d = 0$
g
$\theta(0) = -\pi/2$

Setpoint PID control, $g \neq 0$

$$\tau = K_p\theta_e + K_i \int \theta_e(\mathrm{t})d\mathrm{t} + K_d\dot{\theta}_e$$

$$\tau = M\ddot{\theta} + \mathfrak{m}gr\cos\theta + b\dot{\theta}$$

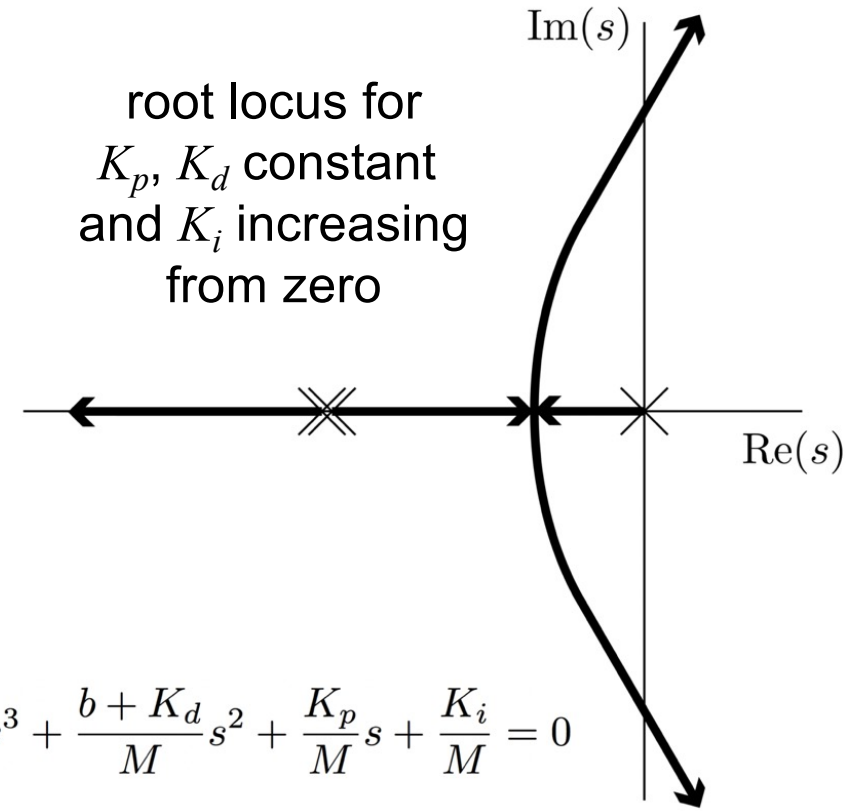
$$\tau_{\text{dist}}$$

$$\dot{\theta}_d = \ddot{\theta}_d = 0$$

$$M\ddot{\theta}_e + (b + K_d)\dot{\theta}_e + K_p\theta_e + K_i \int \theta_e(\mathrm{t})d\mathrm{t} = \tau_{\text{dist}}$$

$$M\theta_e^{(3)} + (b + K_d)\ddot{\theta}_e + K_p\dot{\theta}_e + K_i\theta_e = 0$$

$$s^3 + \frac{b + K_d}{M}s^2 + \frac{K_p}{M}s + \frac{K_i}{M} = 0$$

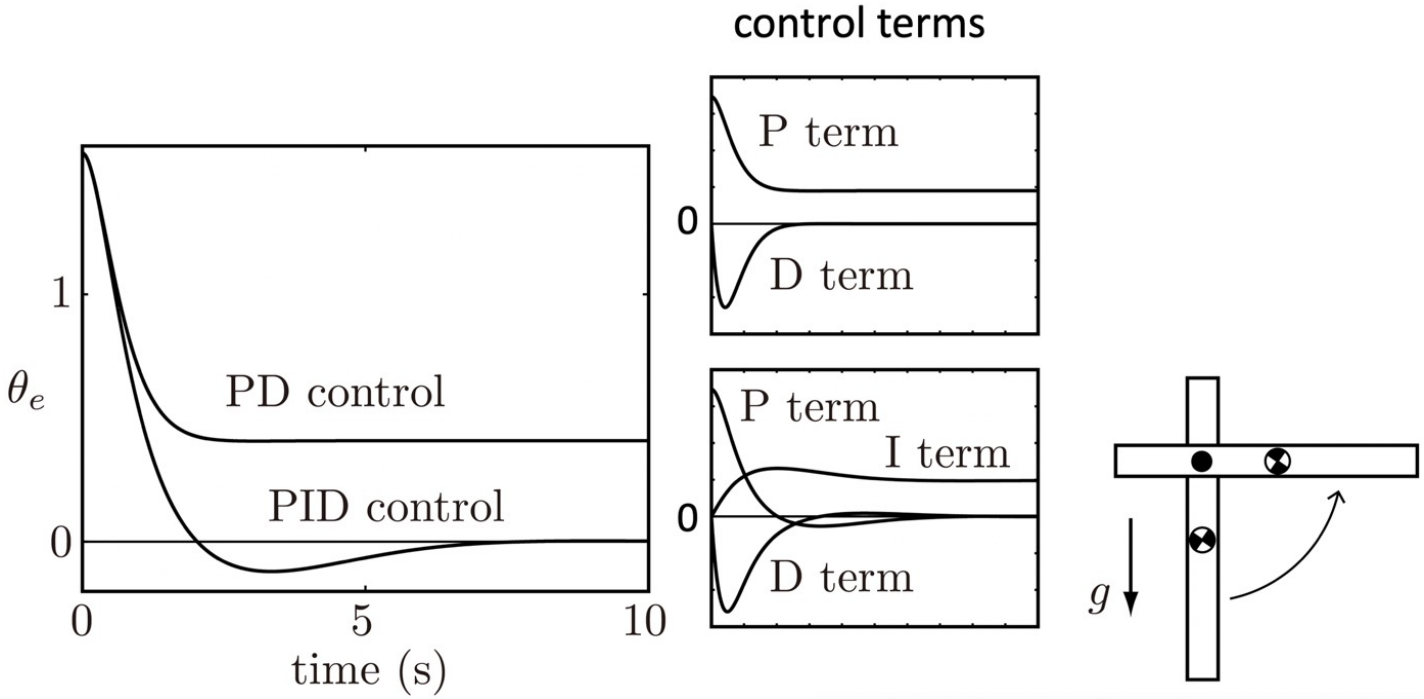


root locus for K_p, K_d constant and K_i increasing from zero

$$s^3 + \frac{b + K_d}{M}s^2 + \frac{K_p}{M}s + \frac{K_i}{M} = 0$$

$$\begin{aligned} K_d &> -b \\ K_p &> 0 \\ \frac{(b + K_d)K_p}{M} &> K_i > 0 \end{aligned}$$

K_i improves steady-state response but can worsen the transient response.


control terms
P term
0
D term
1
θ_e
PD control
P term
I term
PID control
0
D term
0
5
10
time (s)
g