

6.4210/6.4212
Robotic Manipulation

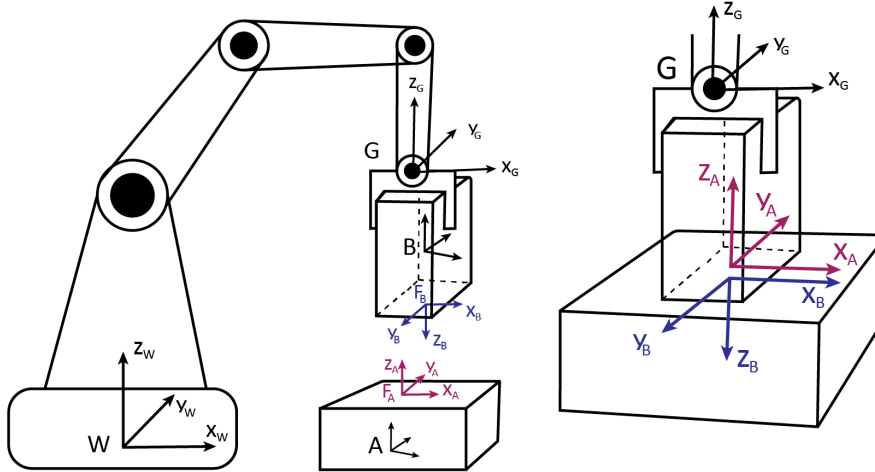
Midterm Practice

Solutions

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1 Frames

- In the figure on the left, the robot is holding block B in its gripper. We want to place the bottom face of B on the top face of block A so that the origin of the face frames coincide and the x axes are aligned (as shown on the right). For each object, we have a frame defined at its center (as shown in the figure) and we know the poses of the face-centered frames for each of the faces (F_A is the frame of the top face on A , and F_B is the frame on the bottom face of B); the z -axis of the face frames points **outward** (as shown in the figure). We also have a frame G defined on the gripper, as shown on the left.



We know: ${}^G X^B$, ${}^W X^A$, ${}^A X^{F_A}$ and ${}^B X^{F_B}$

You can use $T_x(\cdot) = \text{RigidTransform}(\text{MakeXRotation}(\cdot))$ and similarly $T_y(\cdot)$ or $T_z(\cdot)$ in your answers.

- Give an expression for ${}^W X^{G_{place}}$, the rigid transform for the desired gripper placement, as shown on the right side of the figure.

Solution: The origins coincide, the x axes agree, and the outward z axes oppose one another. The right-handed y axes therefore also oppose one another:

$${}^{F_A} X^{F_B} = T_x(\pi).$$

$${}^W X^{G_{place}} = ({}^W X^A)({}^A X^{F_A})({}^{F_A} X^{F_B})({}^{F_B} X^{F_B})^{-1}({}^G X^B)^{-1}$$

2 QP

2. In a problem set, we saw how to keep the end-effector position within a specified "box" in Cartesian coordinates. In this problem, we want to keep the end-effector position near a specified **plane** as well as within a specified box. Imagine that you want to write on a whiteboard or wipe a table. Assume the plane equation is available as a vector of 4 numbers $[n_x, n_y, n_z, d]$ ($\hat{n} = [n_x, n_y, n_z]$ is a unit normal) and the maximum distance allowed from the plane is given as $d_{\max} = 0.01$.

Recall that the signed distance from the point $p = [p_x, p_y, p_z]$ to a plane is $\hat{n} \cdot p + d$; the distance is its absolute value. Assume p , the plane, and the translational rows of J_G are expressed in the same world frame, and that these rows give the velocity of the constrained end-effector point. And remember that the spatial velocity is $\begin{pmatrix} \omega \\ v \end{pmatrix}$.

Here is the key function from the problem set.

```
def DiffIKQP_Wall(
    J_G: np.ndarray,
    V_G_desired: np.ndarray,
    q_now: np.ndarray,
    v_now: np.ndarray,
    p_now: np.ndarray,
) -> np.ndarray:
    prog = MathematicalProgram()
    v = prog.NewContinuousVariables(7, "joint_velocities")
    v_max = 3.0 # do not modify
    h = 4e-3 # do not modify
    lower_bound = np.array([-0.3, -1.0, 0.0]) # do not modify
    upper_bound = np.array([0.3, 1.0, 1.0]) # do not modify

    error = J_G.dot(v) - V_G_desired
    prog.AddCost(error.dot(error))

    prog.AddConstraint(le(v, v_max * np.ones(7)))
    prog.AddConstraint(ge(v, -v_max * np.ones(7)))
    prog.AddConstraint(le(p_now + h * J_G[3:6, :].dot(v), upper_bound))
    prog.AddConstraint(ge(p_now + h * J_G[3:6, :].dot(v), lower_bound))

    solver = SnoptSolver()
    result = solver.Solve(prog)
    if not (result.is_success()):
        raise ValueError("Could not find the optimal solution.")
    v_solution = result.GetSolution(v)
    return v_solution
```

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- (a) What **new** constraints would need to be added to the QP from the problem set? Express your answer as code to add. Assume a variable **plane** is available that holds the plane equation as an array and a variable **d_max** which is a scalar. You do not need to reproduce the whole function; just provide any new constraints.

Solution:

```
p_next = p_now + h * J_G[3:6, :].dot(v)
signed_distance = plane[:3].dot(p_next) + plane[3]
prog.AddConstraint(le(signed_distance, d_max))
prog.AddConstraint(ge(signed_distance, -d_max))
```

- (b) Provide a brief explanation of your answer.

Solution: Distance to **plane** is signed, so to be near the plane requires an upper and lower constraint. We impose $-d_{\max} \leq \hat{n}^T p_{\text{next}} + d \leq d_{\max}$.

3 Grasping

3. (a) We have seen in class that two point contacts with friction are insufficient to guarantee force closure on a three-dimensional body. As in class, we model point contact with no torsional friction. Assuming an antipodal grasp with two grasp points, a high friction coefficient and arbitrary squeezing forces, describe how such a grasp could fail to compensate for an applied force on the body, e.g. gravity.

Solution: The unresisted torque is specifically the component *about the line joining the contacts*; other torque components can be resisted. The object rotates about the contact line. An offset center of mass only causes this failure when gravity has a nonzero moment about that line.

Additional explanation: Put the contacts at $(\pm a, 0, 0)$ and the center of mass at $(0, d, 0)$, with gravity $(0, 0, -mg)$ and $d \neq 0$. Gravity produces $\tau_x = -mgd$. Any force applied at either contact has zero moment about the x axis, so no squeezing force can balance this torque.

- (b) Now, consider the case in which the contacts are "patch" (also called "soft") contacts. Model such a contact as 3 point contacts with friction arranged in a triangle around a contact. This is a simple model of what we get when a (compliant) finger surface contacts an object surface and we have an area of contact. Explain how two antipodal surface areas can resist the applied forces in your counterexample above.

Solution: Tangential friction forces at distinct points in each finite patch can generate a pure torque about the line joining the patch centers.

Additional explanation: For example, opposite vertical forces at opposite lateral offsets create a torque about that line with zero net force. Together with forces supporting the weight, these "couples" (pure torque) can balance the gravity torque in part (a), provided patch size, friction, and normal loading give sufficient moment capacity. A nonzero-area patch can therefore supply the torsional moment that an ideal hard point contact cannot.

- (c) Assume that we want to pick an object in one orientation and place it in a different orientation or place it close to other objects. When choosing antipodal grasps we heuristically check for collisions (with the point cloud) at the pickup point but ignore any collisions at the target pose. Assuming you know the pick pose ${}^W X^{O_{Pick}}$ and the place pose ${}^W X^{O_{Place}}$ how can you modify the point cloud so that the current grasp sampler produces grasps that are collision-free (with respect to the point cloud) for both the pick and place pose?

Solution: Let P_{env} contain the static environment points, with the manipulated object's points removed, and let P_{obj} be its pickup cloud. The rigid motion carrying the object and gripper to placement is

$$D = {}^W X^{O_{Place}} ({}^W X^{O_{Pick}})^{-1}.$$

Pull the placement obstacles back to pickup coordinates and use

$$P_{obstacles} = P_{env} \cup D^{-1} P_{env}, \quad D^{-1} = {}^W X^{O_{Pick}} ({}^W X^{O_{Place}})^{-1}.$$

Sample contacts on P_{obj} and collision-check the pickup gripper against this union, with the sampler's usual object-contact rules.

Additional explanation: Retain the original obstacles: transforming them alone would lose the pickup collision check. Do not duplicate the moving object as a static obstacle. This checks the two endpoints against the available cloud; the transit path still requires collision checking.

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4 RRT

4. (a) Assuming a mobile robot wants to move from one side of a room to the other while avoiding collisions. In which setting do you expect RRT to struggle the most? Circle your answer.
1. A cluttered room with many small obstacles scattered evenly around
 2. A cluttered room with obstacles scattered evenly around and a column in the center.
 3. An open room with a few large obstacles
 4. An open room with a wall in the middle such that there is a single narrow door to get from one side of the room to the other

Briefly explain your answer.

Solution: A narrow door creates a narrow passage in configuration space, with low probability of useful samples and successful extensions through it.

- (b) When searching for efficient paths, we have to beware of "shortest paths" since such paths move very close to obstacles, in fact, touching the obstacles. Consider two strategies for achieving some clearance:
1. run RRT-Connect multiple times, return the path that minimizes $(\text{pathLength} - \alpha * \text{minClearanceAlongPath})$ for some value of α .
 2. run RRT-Connect once but don't add to the tree any samples that are too close to obstacles (using some pre-defined threshold).

Give some pros and cons for both of these strategies; be sure to include the impact on completeness.

Solution: The first strategy is unlikely to produce a path with minimum clearance, but it keeps the probabilistic completeness guarantees of RRT. The second strategy is more likely to produce a path with good clearance but it gives up probabilistic completeness (for finding some path if it exists). It is probabilistically complete for a new problem with expanded C-space obstacles.

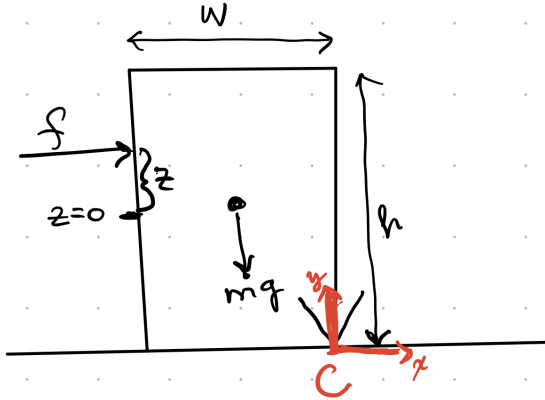
Additional explanation: Strategy 1 prefers shorter paths and larger minimum clearance when $\alpha > 0$, but guarantees neither a specified clearance nor an optimal score. Multiple runs cost more computation. Scoring successful paths does not remove the planner's usual asymptotic probabilistic completeness under its standard assumptions; finitely many bounded runs do not guarantee success.

Strategy 2 seeks a hard clearance threshold δ . Merely rejecting nearby *samples* is insufficient: every connecting edge must also maintain clearance δ . With this edge check, it plans in the restricted free space obtained by expanding configuration-space obstacles. It can be probabilistically complete there under the usual assumptions, but can reject all paths available in the original free space, including narrow passages (or invalidate the start or goal). It offers a clearance guarantee when successful, at the cost of a smaller feasible space and possibly longer paths or more rejections.

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5 Force control

5. Consider a block of height h and width w on a horizontal plane. Its mass is m and the coefficient of static friction is μ . We apply a purely horizontal force of magnitude f on the left side at a height of z relative to the center of mass, that is, at $z + h/2$ above the support surface. Note that $-h/2 < z < h/2$.



We want to understand what applied force will cause the block to slide or pivot around the bottom right corner C . Analyze the onset of motion as the force is increased quasistatically from rest. At incipient tipping, the support resultant acts at the bottom right corner. Use this limiting configuration for the torque calculation; subsequent sliding/tipping dynamics are outside the scope of this problem.

Recall that the (signed) magnitude of the cross-product $a \times b$ for planar vectors is $a_x b_y - a_y b_x$.

- (a) Consider the torques **acting around the bottom right corner** (the potential pivot point).

- i. What is the signed magnitude of the torque due to the applied force f ?

Solution: $-f(h/2 + z)$, taking counterclockwise torque as positive.

- ii. What is the signed magnitude of the torque due to the gravitational force?

Solution: $mgw/2$

- iii. What is the signed magnitude torque due to the contact force (from the contact at the bottom right corner)?

Solution: Zero.

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- (b) What is the condition on f for tipping around the bottom right corner to occur? Hint: This should be a function of z . Write a mathematical expression and explain your reasoning.

Solution: The net torque about C is $mgw/2 - f(h/2 + z)$. The incipient-tipping threshold is $f_{\text{tip}} = mgw/(h + 2z)$; a larger force gives clockwise torque. For tipping about a *sticking* corner to start before sliding, this threshold must be below $f_{\text{slide}} = \mu mg$. Equality gives simultaneous limiting conditions.

- (c) What is the maximum (tangential) force due to friction?

Solution: $|f_t| \leq \mu N = \mu mg$ at quasistatic onset, since the applied force is horizontal. The maximum static-friction magnitude is μmg .

- (d) If we want sliding to begin before tipping as the force is increased from rest, how should we pick z (the height of the applied force)?

Solution:

For $\mu > 0$, require the sliding threshold to be smaller:

$$\mu mg < \frac{mgw}{h + 2z} \iff z < \frac{1}{2} \left(\frac{w}{\mu} - h \right), \quad -h/2 < z < h/2.$$

Thus apply the force sufficiently low.

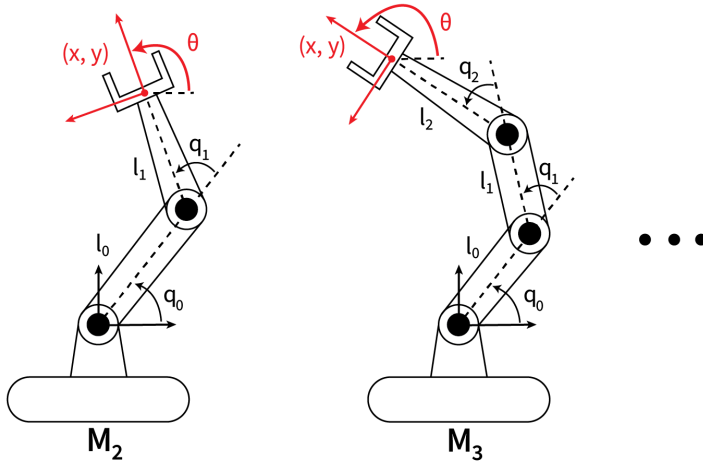
Additional explanation: A force just above μmg and below $mgw/(h + 2z)$ starts sliding before the tipping threshold is reached. Equality of the thresholds does not guarantee sliding alone. If $\mu = 0$, any sufficiently small positive force starts sliding first. These onset conditions do not guarantee no tipping during later accelerated motion.

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6 Jacobian

6. Consider a family of n link planar robots M_n (see the figure):

- M_2 : a 2-link manipulator with joint angles q_0, q_1 , end-effector position x, y and end-effector orientation θ .
- M_3 : a 3-link manipulator with joint angles q_0, q_1, q_2 , end-effector position x, y and end-effector orientation θ .
- etc.



The forward kinematics of this class of robots can be written as:

$$\theta = q_{0:n-1}, \quad x = \sum_{i=0}^{n-1} l_i \cos(q_{0:i}), \quad y = \sum_{i=0}^{n-1} l_i \sin(q_{0:i}),$$

where $q_{0:k} = \sum_{i=0}^k q_i$. Let G be the end-effector (gripper) frame, whose center is at x, y in the world frame and whose orientation is θ in the world frame.

(a) Now, consider two alternative representations for the orientation of the gripper frame with respect to the world frame:

1. θ - a scalar angle
2. $(\cos(\theta), \sin(\theta))$ - a tuple of scalars between -1 and 1

We will refer to the **analytic** Jacobians for the first and second representation as $J_{a1,n}^G$ and $J_{a2,n}^G$, respectively, and the **geometric** Jacobians as $J_{g1,n}^G$ and $J_{g2,n}^G$, where n refers to links as in M_n .

Use the reduced planar geometric velocity $V^G = (\dot{\theta}, \dot{x}, \dot{y})^T$, expressed in world coordinates at G ; G identifies the moving frame, not the expression frame. A full 3D spatial Jacobian would instead have six rows. Use analytic coordinates $(\theta, x, y)^T$ and $(\cos \theta, \sin \theta, x, y)^T$, respectively. The orientation-first ordering follows the Drake convention.

Hint: The matrix E requested below will have this structure: $\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$, for some unknown matrix M (which will depend on θ), so you don't need to compute $\partial x / \partial q$ and $\partial y / \partial q$. Recall that the derivative of $\sin(x)$ is $\cos(x)$ and the derivative of $\cos(x)$ is $-\sin(x)$.

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- i. (2 points) What is the shape of the **geometric** $J_{g1,n}^G$ matrix for the scalar representation of rotation?

Solution: $3 \times n$

- ii. (2 points) What is the shape of the **analytic** $J_{a1,n}^G$ matrix for the scalar representation of rotation?

Solution: $3 \times n$

- iii. (3 points) Write the matrix E such that $J_{a1,2}^G = EJ_{g1,2}^G$.

Solution: $E = I_3$, because planar angular velocity is $\dot{\theta}$.

- iv. (2 points) What is the shape of the **geometric** $J_{g2,n}^G$ matrix for the tuple representation of rotation?

Solution: $3 \times n$

- v. (2 points) What is the shape of the **analytic** $J_{a2,n}^G$ matrix for the tuple representation of rotation?

Solution: $4 \times n$

- vi. (3 points) Write the matrix E such that $J_{a2,2}^G = EJ_{g2,2}^G$.

Solution:

$$E = \begin{bmatrix} -\sin(\theta) & 0 & 0 \\ \cos(\theta) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

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- (b) Assume a non-singular configuration of the manipulator and a commanded spatial velocity $V^G = [\dot{\theta}, \dot{x}, \dot{y}]$. Let:

$$v_n = [J_n^G]^+ V^G$$

and

$$\hat{V}_n^G = J_n^G v_n$$

where n indicates the number of links and $[\cdot]^+$ denotes the pseudo-inverse. We'll assume geometric Jacobians for a scalar rotation representation, i.e. $J_n^G = J_{g1,n}^G$ from the previous part.

Assume $\text{rank}(J_n^G) = \min(3, n)$. For $n = 2$, take a command V^G outside the column space of J_2^G ; this makes the intended unachievable-command case explicit. For $n = 3, 4$, the command is arbitrary. Select all true statements under these assumptions:

1. $\hat{V}_n^G = V^G$
 2. $\hat{V}_n^G \neq V^G$, and v_n is the unique minimizer of $\min_v \|J_n^G v - V^G\|^2$.
 3. v_n is the unique value for which $J_n^G v = V^G$
 4. There are infinitely many v satisfying $J_n^G v = V^G$; $v_n = (J_n^G)^+ V^G$ is the unique minimum-norm member of this set.
- i. (2 points) For $n = 2$ which of these statements hold? Explain briefly.

Solution: Statement 2. The 3×2 full-column-rank Jacobian cannot produce this command. Its pseudoinverse gives the unique least-squares velocity, and $\hat{V}_2^G$ is the orthogonal projection onto its column space. If the command were achievable, statements 1 and 3 would hold instead.

- ii. (2 points) For $n = 3$ which of these statements hold? Explain briefly.

Solution: Statements 1 and 3. The 3×3 Jacobian is invertible, so $v_3 = (J_3^G)^{-1} V^G$ achieves the command and is the only solution.

- iii. (2 points) For $n = 4$ which of these statements hold? Explain briefly.

Solution: Statements 1 and 4. The 3×4 Jacobian has rank three and nullity one. Every command is achievable, with solutions $v = v_4 + z$, $z \in \ker J_4^G$. The pseudoinverse selects a unique minimum-norm solution; it is the exact-solution set that is nonunique.