

Problem Set 4: Grasping

6.4210/2 Robotic Manipulation — Fall 2026

Due: October 11, 2026

Out: October 1 Due: October 11

This problem set is about contact and grasping: when a contact holds without slipping, how to estimate surface normals from a depth image, and how to use those normals — analytically and by sampling — to find antipodal grasps.

Before you start: update your Python environment. The ps4 code builds letter meshes, which needs packages that earlier ps4s did not. Pull the latest handout and check the install instructions in its `README.md` (§3.3). If you made your venv before ps4, activate it and run

```
uv pip install "manipulation[letter]" scipy rtree
```

On Colab, `setup()` installs them for you.

1 Bookwork

1.1 Contact models and wrenches

Use the notation of Chapter 5. A spatial force $F_{i,W}^{B_p} = [(\tau_{i,W}^{B_p})^T, (f_{i,W}^{B_p})^T]^T$ has *torque first*, is applied at point B_p of body B , and is expressed in W . A contact frame C_i has $+z$ pointing into the object, so positive $f_{C_i,z}^{C_i}$ is a push. Surface normals used in the coding problems point *out* of the object; keep this distinction in mind. Enter the answers in the **Gradescope** blocks in Gradescope. The **Going further** prompts are for discussion and require no submission.

Problem 1 What can a contact transmit?

[3]

Gradescope:

1.1 Match each model to a description below. Use each letter once.

[1]

Models

- i. Frictionless point contact: C
- ii. Point contact with friction: D
- iii. Soft-finger contact: A
- iv. Hydroelastic contact with friction: B

Descriptions

- A. A nonnegative normal force, tangential force in a friction cone, and a bounded torsional moment about the contact normal.
- B. Contact forces over a compliant contact patch, integrated to obtain a spatial force.
- C. A nonnegative normal force only, with no moment at the contact point.
- D. A nonnegative normal force and tangential force in a friction cone, with no moment at the contact point.

Solution: *i–C, ii–D, iii–A, iv–B. In every model the contact pushes rather than pulls, so the normal component $f_{C_i,z}^{C_i}$ (along $C_{i,z}$, perpendicular to the surface) is nonnegative.*

- **i. Frictionless point contact** → **C**. It transmits only this normal force.
- **ii. Point contact with friction** → **D**. Friction adds tangential components, in the surface, but a force applied at a single point still has no moment about that point: recall that the moment about a point is $r \times f$, and here $r = 0$.
- **iii. Soft-finger contact** → **A**. It adds a bounded twisting moment about the normal, representing a fingertip that touches over a small patch rather than a point.
- **iv. Hydroelastic contact with friction** → **B**. Drake's compliant model treats both bodies as slightly deformable and integrates the pressure over the patch: its net force and moment depend on both the pressures and where on the patch they act.

1.2 For point contact with friction, select all constraints that must hold:

[1]

- ☒ $f_{C_i,z}^{c_i} \geq 0$.
- ☒ $\sqrt{(f_{C_i,x}^{c_i})^2 + (f_{C_i,y}^{c_i})^2} \leq \mu f_{C_i,z}^{c_i}$.
- ☐ The tangential force always has magnitude $\mu f_{C_i,z}^{c_i}$.
- ☒ $\tau_{i,C_i}^{c_i} = 0$.

Solution: The first, second, and fourth constraints hold.

- $f_{C_i,z}^{c_i} \geq 0$: **holds**. Unilateral contact: the finger can push but not pull.
- The **square-root inequality**: **holds**. It is the friction cone, which limits tangential friction to μ times the normal force.
- **Tangential force always $\mu f_{C_i,z}^{c_i}$** : **false**. While the contact is sticking (not sliding), the tangential force can be anywhere inside the cone, taking the value needed for equilibrium. Equality holds only when the contact is about to slide.
- $\tau_{i,C_i}^{c_i} = 0$: **holds**. An ideal point contact applies no moment at its own location; moments about other points arise from the force's moment arm.

1.3 True or false: because a point contact cannot apply a moment at its own location, it cannot cause angular acceleration of the object. ☐ True ☒ **False**

[1]

Solution: False. Angular acceleration depends on the moment about the center of mass B (the name Problem 2 also uses), not about the contact. Shifting the contact wrench to B , i.e. finding the equivalent force and moment there, leaves the force unchanged and adds the force's lever-arm moment:

$$\tau_{i,W}^B = \tau_{i,W}^{c_i} + {}^B p_W^{c_i} \times f_{i,W}^{c_i},$$

where ${}^B p_W^{c_i}$ is the contact's position relative to B , expressed in W . The first term vanishes for a point contact, but the second generally does not. For example, an upward force applied to the left of the center of mass creates a moment about W_y . Only a force whose line of action (the line through its point of application, along its direction) passes through the center of mass has zero moment about that point.

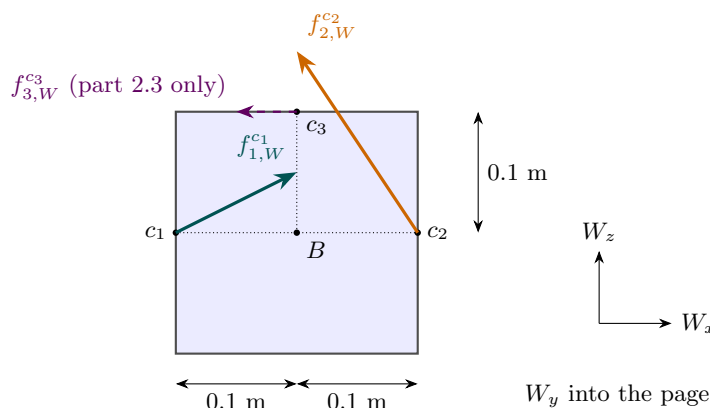
Going further: What would you want to measure before choosing a contact model for a glass marble, a rubber fingertip, or a suction cup?

Solution: The friction coefficient, how much each surface deforms under load (and so the size of the contact patch), and whether the contact can pull. A glass marble on a hard finger is stiff with a tiny patch: a point contact, perhaps nearly frictionless. A rubber fingertip spreads into a patch, so it can also resist twisting: a soft-finger or hydroelastic model, with its stiffness and torsional friction measured. A suction cup breaks the assumption shared by every model in 1.1, $f_{C_i,z}^{c_i} \geq 0$: it can pull, up to a limit set by the seal and the vacuum, which is what to measure.

Problem 2 Forces, moments, and motion

[3]

Consider a planar body moving in the world xz plane. Its center of mass is B , and the only possible rotation is about W_y . A planar spatial force keeps the components $F = [\tau_y, f_x, f_z]^T$, torque first. All data below are expressed in W ; positions are relative to B . Ignore gravity in this question. Positions and forces are given as (x, z) components; all y components are zero. Two point contacts are at $(-0.1, 0)$ and $(0.1, 0)$ meters, and apply pure forces $f_{1,W}^{c_1} = (2, 1)$ and $f_{2,W}^{c_2} = (-2, 3)$ newtons.

**Gradescope:**

- 2.1 Enter $[\tau_{1,W,y}^B, \tau_{2,W,y}^B, \tau_{\text{total},W,y}^B]$ in N m, and the force components $[f_{\text{total},W,x}^B, f_{\text{total},W,z}^B]$ of $F_{\text{total},W}^B$ in N. [1]

$$[\tau_{1,W,y}^B, \tau_{2,W,y}^B, \tau_{\text{total},W,y}^B] = [\underline{0.1}, \underline{-0.3}, \underline{-0.2}] \text{ N m},$$

$$[f_{\text{total},W,x}^B, f_{\text{total},W,z}^B] = [\underline{0}, \underline{4}] \text{ N}.$$

Solution: The answers are $(0.1, -0.3, -0.2)$ N m and $(0, 4)$ N. Recall that the moment of f about B is $p \times f$, with p measured from B ; for a planar force, its y component is $\tau_y = p_z f_x - p_x f_z$. Thus

$$\tau_{1,W,y}^B = 0(2) - (-0.1)(1) = 0.1, \quad \tau_{2,W,y}^B = 0(-2) - (0.1)(3) = -0.3.$$

Adding gives $\tau_{\text{total},W,y}^B = -0.2$ N m. The forces add componentwise: $f_x = 2 - 2 = 0$ and $f_z = 1 + 3 = 4$ N. All moments were taken about the same point B before adding.

- 2.2 The body starts at rest. Does its center of mass initially accelerate, does it initially have angular acceleration, both, or neither? [1]

Answer (translation only / rotation only / both / neither): Both.

Solution: Both. Newton's law gives $ma_{B,x} = 0$ and $ma_{B,z} = 4$, so the center of mass initially accelerates upward. The planar rotational equation is $I_{B,yy}\ddot{\theta} = \tau_{\text{total},W,y}^B = -0.2$, with $I_{B,yy}$ the moment of inertia about the y axis through B , so the angular acceleration is also nonzero for a finite positive moment of inertia. These equations determine initial accelerations; the later motion also depends on how the contact forces and their locations evolve.

- 2.3 A third contact at $(0, 0.1)$ meters applies $f_{3,W}^{c3} = (-1, 0)$ N. Enter the new total spatial force $F_{\text{total},W}^B = [\tau_y, f_x, f_z]^T$. [1]

$$F_{\text{total},W}^B = [\underline{-0.3}, \underline{-1}, \underline{4}]^T \text{ (N m, N, N)}.$$

Solution: The third force contributes

$$\tau_{3,W,y}^B = (0.1)(-1) - 0(0) = -0.1 \text{ N m}.$$

Therefore the new moment is $-0.2 - 0.1 = -0.3$ N m, and the net force is $(0, 4) + (-1, 0) = (-1, 4)$ N in xz coordinates. Stacking torque first gives $F_{\text{total},W}^B = (-0.3, -1, 4)^T$.

Going further: How would you choose contact locations if translation were useful but rotation were undesirable? What additional information would you need to predict the motion a second later, rather than just the initial acceleration?

Solution: Place the contacts so that the net force's line of action passes through B ; for example, push through the center of mass, or use a symmetric pair whose moments about B cancel, as $\tau_y = p_z f_x - p_x f_z$ shows. A second later the body has moved, so we need its mass and moment of inertia, and how the contact forces and their locations change as it moves: whether the fingers follow it, whether the contacts stick or slide, and what else (gravity, a table) acts on it.

1.2 Friction and equilibrium

Problem 3 A box on an inclined plane

[3]

This is based on Exercise 5.1a in the book. A box of mass m rests on a plane inclined at angle γ , as in Figure 1. Assume $0 \leq \gamma < \pi/2$, and that the box does not tip. The resultant plane reaction acts at a center of pressure C somewhere along the base; its line of action passes through the center of mass in equilibrium. Write $f_{C,z}^{BC}$ for its normal component and $f_{C,x}^{BC}$ for its up-slope tangential component. These are sums over the contact patch.

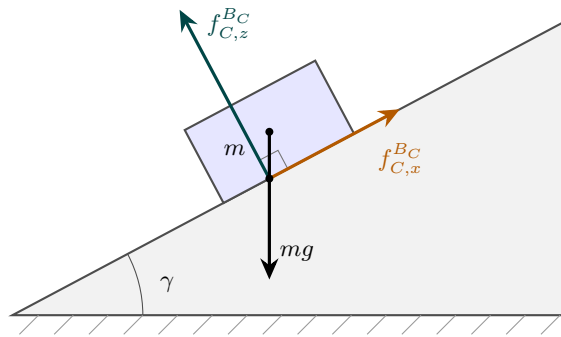


Figure 1: A box of mass m at rest on a plane inclined at γ . Gravity acts at the center of mass; the plane's reaction acts at the center of pressure, and is resolved into the normal component $f_{C,z}^{BC}$ and the tangential (friction) component $f_{C,x}^{BC}$.

Gradescope:

3.1 What is the expression for the friction force $f_{C,x}^{BC}$?

[1]

- A. mg
- B. $mg \cos \gamma$
- C. $mg \sin \gamma$

Solution: Resolve gravity into the contact frame (C_x up the slope, C_z normal to it). Its component along C_x is $-mg \sin \gamma$. Static force balance along that axis requires

$$f_{C,x}^{BC} - mg \sin \gamma = 0, \quad f_{C,x}^{BC} = mg \sin \gamma.$$

The positive sign means that friction acts up the slope. This is the friction force required for sticking; whether friction can supply that much depends on μ , which part 3.3 checks.

3.2 What is the expression for the normal force $f_{C,z}^{BC}$?

[1]

- A. mg
- B. $mg \cos \gamma$
- C. $mg \sin \gamma$

Solution: Gravity's component along the outward plane normal C_z is $-mg \cos \gamma$. With no acceleration normal to the plane,

$$f_{C,z}^{BC} - mg \cos \gamma = 0, \quad f_{C,z}^{BC} = mg \cos \gamma.$$

This is the total normal reaction over the base. Its distribution must also balance moments; the no-tipping assumption ensures that the required center of pressure remains within the base.

3.3 For a given angle γ , what is the minimum coefficient of friction required for the box to not slip down the plane? [1]

- A. $\mu \geq 0$
- B. $\mu \geq \sin \gamma$
- C. $\mu \geq \cos \gamma$
- D. $\mu \geq \tan \gamma$
- E. $\mu \geq \cot \gamma$

Solution: Sticking requires $|f_{C,x}^{BC}| \leq \mu f_{C,z}^{BC}$. Substituting the required force components yields

$$mg \sin \gamma \leq \mu mg \cos \gamma \iff \mu \geq \tan \gamma,$$

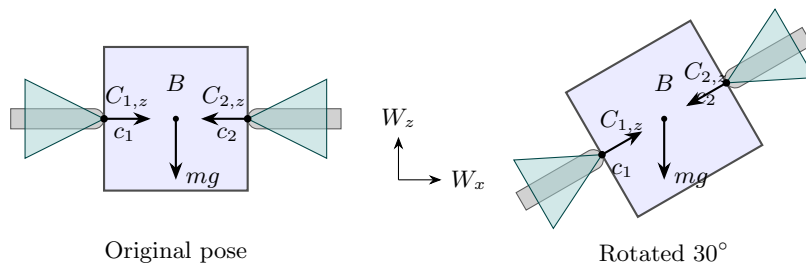
since $mg > 0$ and $\cos \gamma > 0$. Hence $\mu_{\min} = \tan \gamma$. At equality the contact is at its friction limit, and any steeper slope makes the box slide. Geometrically, recall that the friction cone has half-angle $\arctan \mu$ about the normal. The reaction must be vertical to cancel gravity, which puts it at angle γ from the slope's normal, so it lies in the cone exactly when $\gamma \leq \arctan \mu$. This sliding condition does not by itself rule out tipping.

Going further: How would you decide whether a tall box will slide or tip first? What might change if the base were compliant, or if the friction coefficient varied along the ramp? Which measurements would help distinguish these effects?

Solution: Let the base have half-width b and the center of mass height h above it. The box slides once $\tan \gamma > \mu$, and tips once gravity's line of action passes the downhill edge, $\tan \gamma > b/h$; whichever threshold is smaller happens first, so a tall box ($b/h < \mu$) tips. On a compliant base the downhill edge sinks, and the center of pressure moves downhill before the box visibly tips. If μ varies, sliding starts wherever friction is lowest. To tell these apart: tilt the ramp slowly and watch whether the box translates or rotates about its edge; measure μ separately with a low, flat block; and measure where the base pressure sits, for instance with a pressure-sensitive mat or a force plate.

Problem 4 Holding a block between two fingers [5]

A square of side $2a = 0.2$ m has its center of mass at its center B and weight $mg = 1$ N. Work in the xz plane, with W_z upward; positions and forces are given as (x, z) components. Ideal point fingers touch the midpoints of the left and right faces, with $\mu = 1/2$. The green friction cones are drawn *outward* from the object, along its outward surface normals. They represent admissible (consistent with the friction cone) forces on the fingers; the corresponding forces on the object point in the opposite directions. Arrows labeled $C_{i,z}$ show the inward contact-frame normals used in the equations. Gray shapes depict fingers; treat their tips as point contacts.



Gradescope:

- 4.1 What is the full opening angle of each friction cone, in degrees to two decimal places? 53.13°. [1]

Solution: Let α be the angle from the cone axis to one edge. At the edge, $|f_t| = \mu f_n$, so $\tan \alpha = |f_t|/f_n = \mu$. The full opening includes both edges:

$$2\alpha = 2 \arctan(\mu) = 2 \arctan(1/2) = 53.13^\circ.$$

Drawing the cone outward rather than inward reverses its axis, but does not change its opening angle.

- 4.2 In the original pose, each finger supplies a *normal component* of exactly 1 N; its tangential component is whatever equilibrium requires. Can the grasp balance gravity? Enter yes/no and the upward tangential force at each finger. Is the solution strictly inside the friction cones or on their boundaries? [1]

Balance gravity (yes/no): Yes.

Upward tangential force: left 0.5 N; right 0.5 N.

Cone location (interior/boundary): Boundary.

Solution: Yes. Write the upward tangential (friction) components as t_1, t_2 . The equal opposing normal components cancel horizontally. Vertical balance and moment balance about B , with $a = 0.1$ m, require

$$t_1 + t_2 = mg = 1, \quad at_1 - at_2 = 0.$$

Thus $t_1 = t_2 = 0.5$ N. Each friction bound is $|t_i| \leq \mu f_{C_i,z}^c = (1/2)(1) = 0.5$ N. Both hold with equality, so both contacts are exactly on their cone boundaries (friction at its maximum). Equilibrium is possible, but there is no remaining upward friction capacity to resist an extra downward load.

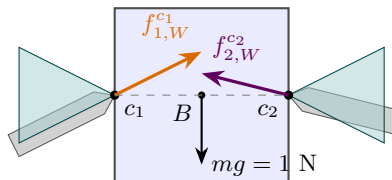
- 4.3 Now rotate the square and its two midpoint contacts counterclockwise by 30° in the drawing. The contact normals rotate with the faces. Each normal component is still exactly 1 N. Can these contacts balance gravity? (yes/no): No. [1]

Solution: No. In xz coordinates let $\beta = 30^\circ$, $u = (\cos \beta, \sin \beta)$ point from the left contact to the right, and $v = (-\sin \beta, \cos \beta)$ point along the rotated faces. The two normal forces are u and $-u$ N and cancel. The remaining contact force is $(t_1 + t_2)v$. Balancing gravity would require

$$-(t_1 + t_2) \sin \beta = 0, \quad (t_1 + t_2) \cos \beta = 1.$$

The first equation forces $t_1 + t_2 = 0$, contradicting the second. Therefore force balance already fails, even before checking moments or friction. With independently adjustable normal forces, equilibrium instead requires $N_1 - N_2 = \sin \beta = 0.5$ N and $t_1 = t_2 = \cos \beta / 2 = \sqrt{3}/4$ N. For example, $N_2 = \sqrt{3}/2$ N and $N_1 = (1 + \sqrt{3})/2$ N satisfy both friction bounds. The failure above is due to the fixed normal forces, not a loss of planar force closure (the ability to resist every load when the normal forces are free to change; see Problem 5).

- 4.4 Return the block to the original pose. The left finger's total force is fixed at $f_{1,W}^{c_1} = (1, 0.5)$ N, while the right finger's total force is $f_{2,W}^{c_2} = (-1, 0.25)$ N. These are asymmetric force directions, *not* rotated contact normals. Are both forces admissible? Enter the total spatial force $F_{\text{total},W}^B = [\tau_y, f_x, f_z]^T$ about B , including gravity, and say whether static equilibrium holds. [1]



Arrows: forces on the object

Both forces admissible (yes/no): Yes.

$$F_{\text{total},W}^B = [\underline{0.025}, \underline{0}, \underline{-0.25}]^T \quad (\text{N m}, \text{N}, \text{N}).$$

Static equilibrium (yes/no): No.

Solution: Both forces are admissible (inside their friction cones): each has normal component 1 N, and their tangential magnitudes satisfy $0.5 \leq 0.5$ and $0.25 \leq 0.5$ N. The left force is at its friction limit; the right force is inside its cone. Including gravity, force balance gives

$$f_x = 1 - 1 = 0, \quad f_z = 0.5 + 0.25 - 1 = -0.25 \text{ N}.$$

The contact locations are $(-a, 0)$ and $(a, 0)$, so

$$\tau_y = a(0.5) - a(0.25) = 0.025 \text{ N m}.$$

Gravity acts at B and contributes no moment about B . Hence $F_{\text{total},W}^B = (0.025, 0, -0.25)^T$; both vertical force and moment balance fail. Admissible individual forces need not produce an equilibrium.

- 4.5 With frictionless contacts at these same two face midpoints, can any choice of compressive normal forces hold the block against gravity? (yes/no): No. [1]

Solution: No. With $\mu = 0$, the cone inequality forces each tangential component to zero. The two contact forces are therefore horizontal, $(N_1, 0)$ and $(-N_2, 0)$, with $N_i \geq 0$. Changing their magnitudes can change the horizontal resultant, but the total vertical force remains $-mg = -1$ N. No amount of squeezing in this geometry balances gravity.

Going further: Which would you change first to make this grasp reliable: squeeze force, contact locations, fingertip material, or sensing? How would your choice change for a fragile object? Would changing the finger approach directions change the contact normals if the contacts stay on the same flat faces?

Solution: In 4.2 both contacts sit exactly on their cone boundaries, so the grasp has no margin. Squeezing harder adds margin, since the friction available grows with the normal force. Higher-friction fingertips add margin without more force, and so does moving the contacts so that the grasp line points through the center of mass. For a fragile object, squeeze force is the last thing to raise: prefer high-friction, compliant fingertips, and slip sensing so the squeeze stays near the minimum needed. Approach direction does not change the normals: on a flat face, the contact normal is the face's normal. It only changes which force in the cone the finger tries to apply.

Problem 5 Force closure is more than holding against gravity [3]

This problem continues Problem 4: the same square block, held at the midpoints of its left and right faces by two point fingers with $\mu = 1/2$. Work in the same vertical xz plane, with translation in x and z and rotation about W_y . A planar spatial force has components $F = [\tau_y, f_x, f_z]^T$. Let $\mathcal{K}_{\text{total},W}^B$ be the Minkowski sum of the contact wrench cones, all shifted to B and expressed in W . Force closure means that this cone is the entire planar wrench space $\mathbb{R}^3$. For this idealized definition, normal forces can grow without bound and each finger can independently choose an admissible force. A real force limit instead bounds the disturbances that can be resisted.

Gradescope:

- 5.1 Does the original-pose grasp from Problem 4 have *planar* force closure when the fixed 1 N normal-force restriction is removed? (yes/no): Yes. [1]

Solution: Yes. Recall that a contact's wrench cone is the set of wrenches about B it can apply, one per force in its friction cone, so their Minkowski sum is every total wrench the two fingers can apply together. To show that this is all of $\mathbb{R}^3$, construct any desired planar contact spatial force $F_{\text{contact},W}^B = [\tau_y, f_x, f_z]^T$. Write the left and right forces in xz coordinates as (N_1, t_1) and $(-N_2, t_2)$. Their net wrench satisfies

$$f_x = N_1 - N_2, \quad f_z = t_1 + t_2, \quad \tau_y = a(t_1 - t_2).$$

Choose

$$t_1 = \frac{f_z + \tau_y/a}{2}, \quad t_2 = \frac{f_z - \tau_y/a}{2}, \quad N_1 = N + \frac{f_x}{2}, \quad N_2 = N - \frac{f_x}{2}.$$

Because $\mu > 0$, taking $N \geq |f_x|/2 + \max(|t_1|, |t_2|)/\mu$ makes both normal forces nonnegative and ensures $|t_i| \leq \mu N_i$. Thus every planar wrench can be produced, including the opposite of any external load. The construction uses the assumption of unbounded squeeze force.

- 5.2 In the plane, for each contact model below, can force closure be achieved with *some suitable object and contact placement* using two, three, or four contacts? Enter a row of three yes/no answers for each model. Additional contacts may apply zero force; this is an existence question, not a guarantee for arbitrary placements. [1]

Contact model	Two contacts	Three contacts	Four contacts
With friction ($\mu > 0$)	<u>Yes</u>	<u>Yes</u>	<u>Yes</u>
Frictionless	<u>No</u>	<u>No</u>	<u>Yes</u>

Solution: With friction: yes, yes, yes. The two-contact construction above already achieves closure; additional contacts can apply zero force. Frictionless: no, no, yes. A frictionless contact can only push along its normal, so it supplies one wrench ray $\lambda_i w_i$ with $\lambda_i \geq 0$.

- **Two contacts: no.** Planar wrenches are three-dimensional, and two rays cannot reach every direction.
- **Three contacts: no.** Three independent rays still generate only a cone: their multipliers cannot reverse sign, so each $-w_i$ is out of reach.
- **Four contacts: yes.** Four suitably placed contacts can positively span the space (reach every wrench with nonnegative multipliers), which takes at least one more ray than the dimension.

For example, on a square take inward normal contacts at $(x, z) = (-a, b)$, $(a, 0)$, $(b, -a)$, and $(0, a)$, with $0 < b < a$. Their spatial forces $[\tau_y, f_x, f_z]^T$ per unit normal force are

$$(b, 1, 0)^T, \quad (0, -1, 0)^T, \quad (-b, 0, 1)^T, \quad (0, 0, -1)^T.$$

They span $\mathbb{R}^3$ and sum to zero. Any real-coefficient representation of a wrench can therefore be made nonnegative by adding the same sufficiently large number to all four coefficients, without changing the wrench.

- 5.3 Four frictionless fingers touch the four face midpoints of a square. Does this arrangement have planar force closure? (yes/no): No. [1]

Solution: No. At each face midpoint the position vector from B and the contact normal are collinear. Thus ${}^B p_W^{c_i} \times f_{i,W}^{c_i} = 0$ for every contact, regardless of its force magnitude. Every attainable planar wrench consequently has $\tau_y = 0$, so a nonzero external couple (a pure moment, with no net force) cannot be opposed. Four is a possible contact count for closure, but these particular placements do not achieve it.

Going further: For what tasks would you accept a grasp without force closure? How might a table, gravity, or a second hand help? What does the ideal force-closure test leave out about how a real hand generates and controls contact forces?

Solution: When the loads are known and small: carrying a tray level, sliding a box across a table, or setting an object down. A table supplies contacts that resist downward and sideways loads, gravity is a known load that keeps the object pressed into those contacts, and a second hand adds contacts on the other side. The ideal test assumes unbounded normal forces, and that each finger can choose any force in its cone. A real hand has finite joint torques, compliant links, and kinematics that determine which contact forces it can produce, and how fast it can change them. It also has uncertain friction and contact locations, and it must apply those forces while moving.

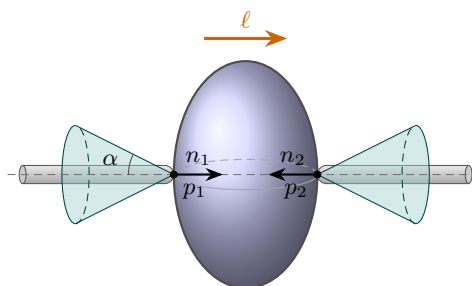
Problem 6 Antipodal grasps and force closure

[3]

Two fingers touch a rigid object in 3D at p_1 and p_2 . The inward surface normals are n_1 and n_2 , and the friction angle is $\alpha = \arctan \mu$. Let ℓ be the unit vector from p_1 to p_2 . The grasp is *antipodal* when the line connecting the contacts lies inside both friction cones:

$$\angle(\ell, n_1) \leq \alpha \quad \text{and} \quad \angle(-\ell, n_2) \leq \alpha.$$

This lets the fingers apply equal and opposite squeezing forces along the line without slipping at the contacts. For this question, assume the line lies strictly inside both cones. As in the earlier figures, the green cones point outward; the normals n_i point into the object.



A spatial two-finger grasp

Gradescope:

6.1 With hard point contacts and friction, which wrench can the grasp never resist, no matter how hard the fingers squeeze or how large μ is? [1]

- A. A force along ℓ
- B. A force perpendicular to ℓ
- C. A torque about ℓ
- D. A torque about an axis perpendicular to ℓ

Solution: A torque about ℓ . Put the origin on the grasp line (through p_1 and p_2), so $p_i = t_i \ell$. Recall that a moment about an axis is the component of $p \times f$ along it; for any contact force f_i ,

$$\ell \cdot (p_i \times f_i) = t_i \ell \cdot (\ell \times f_i) = 0.$$

The last step uses $\ell \times f_i \perp \ell$. Neither force can produce a moment about the grasp line, so nothing can resist an external twist about it.

6.2 What is the minimum number of hard point contacts with friction needed for force closure in 3D? 3 [1]

Solution: Three. Any two contacts lie on a line and, by part 6.1, cannot resist torque about it. Three contacts that are not on one line can provide force closure with suitable locations, normals, and friction.

6.3 Suppose each contact is a soft finger and can also apply a moment about its normal, with $|\tau_i| \leq \mu_t f_{n,i}$ and $\mu_t > 0$. Does this antipodal grasp have force closure? (yes/no): Yes. [1]

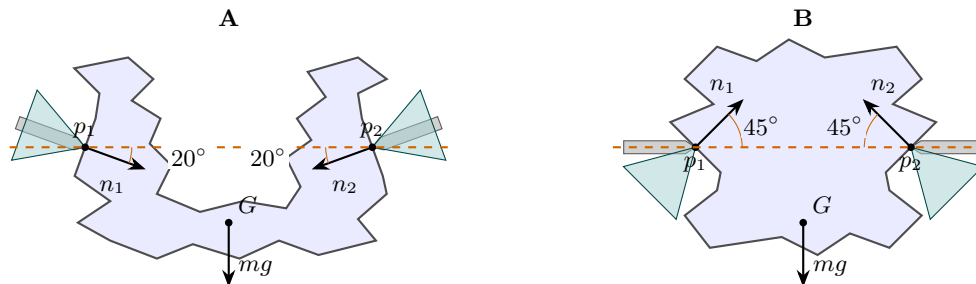
Solution: Yes. Because ℓ lies strictly inside both friction cones, the two contact forces can resist forces in all directions and moments perpendicular to ℓ . The only gap was torque about ℓ (part 6.1). The soft fingers fill it: in an antipodal grasp each normal is close to $\pm \ell$, so each finger's twisting moment about its normal has a nonzero component along ℓ , in either direction. Squeezing harder increases the normal force $f_{n,i}$ and so the available friction forces and moments.

Problem 7 Antipodal grasps on non-convex objects

[6]

Consider the two rigid objects below in the vertical xz plane, with $+x$ to the right and $+z$ upward. Each has weight $mg > 0$ and center of mass G directly below the midpoint between the contacts. The contacts are hard point contacts with $\mu = \tan 30^\circ$. Each finger can independently apply any force in its friction cone, with no upper bound on its normal force.

Each contact lies on a flat facet, not at a corner. The black arrows are inward normals, the green cones point outward, and the dashed orange line joins the contacts. All labeled angles are exact.

**Gradescope:**

7.1 Is the contact pair in A antipodal? (yes/no): Yes.

[3]

Solution: Yes. Recall from Problem 6 that a pair is antipodal when the contact line makes at most the friction angle $\alpha = \arctan \mu = 30^\circ$ with each inward normal. Each angle is $20^\circ < 30^\circ$. The segment need not lie entirely inside the object.

a. Can these contacts hold the object in static equilibrium against gravity? (yes/no): Yes.

Solution: Yes. Forces $(S, mg/2)$ and $(-S, mg/2)$ in xz coordinates balance both gravity and moments about G . With $S \geq mg/(2 \tan 10^\circ)$, each force is at most 10° above horizontal and lies in its cone. Here S is the horizontal (squeezing) force component. Each cone extends 30° either side of its normal, and each normal is 20° below horizontal, so the cone reaches 10° above horizontal.

b. Do these contacts have planar force closure? (yes/no): Yes.

Solution: Yes. The contact line lies strictly inside both friction cones, so this two-contact planar grasp has force closure by the same construction as part 5.1: squeezing along the line provides as much normal force as needed, and the room left inside each cone supplies friction in either tangential direction.

7.2 Is the contact pair in B antipodal? (yes/no): No.

[3]

Solution: No. The fingers touch opposite sides at the same height, but each inward normal makes 45° with the line joining the contacts. This exceeds the 30° cone half-angle.

a. Can these contacts hold the object in static equilibrium against gravity? (yes/no): Yes.

Solution: Yes. Choose forces $(mg/2, mg/2)$ and $(-mg/2, mg/2)$ in xz coordinates. Each lies along its inward normal. Their sum is $(0, mg)$ and their moments about G cancel; no tangential friction force is needed.

b. Do these contacts have planar force closure? (yes/no): No.

Solution: No. The left inward normal points 45° above $+x$, and its cone extends 30° either side, so it spans 15° – 75° from $+x$; by symmetry the right cone spans 105° – 165° . Every angle in these ranges points upward, so every nonzero contact force has an upward component. These contacts cannot produce a downward resultant to oppose an upward external force.

Going further: In A, part of the line between the contacts lies outside the object. Does that matter for whether the pair is antipodal? In B, what do the local surface normals tell you that the finger directions do not? Suppose the estimated normals are off by a few degrees. Which grasps would still be

antipodal? How would curved surfaces affect your answer if the fingers landed slightly away from the planned contact points?

Solution: No: being antipodal depends only on the contact line and the two friction cones. The force each finger applies acts along that line whether or not the line stays inside the object (the fingers must still reach the contacts without colliding, which is a separate check). In B the fingers point straight at each other, which looks opposed. The normals show that each facet is tilted 45° , outside the 30° cone, so the fingers would slide. With normals off by a few degrees, A (at 20° , 10° inside the cone) stays antipodal and B (15° outside) stays non-antipodal; a pair near the 30° boundary could flip either way. On a curved surface the normal changes along the surface, so a finger landing off its planned point meets a different normal. The more curved the surface, the smaller the landing error that can push a pair outside its cones.

Problem 8 What should a grasp-quality score measure?

[4]

A grasp planner can produce thousands of candidate grasps, so we want a single number that ranks them. This problem builds one standard score in four steps and then asks what it does and does not measure. Stay in the plane of Problem 5: the vertical xz plane, with translation in x and z and rotation about W_y .

1. From cones to finitely many forces. A friction cone holds infinitely many forces. Replace it with a few *generator* forces along its edges, each scaled to a unit normal component. In the plane, a cone has exactly two edges, so its two edge forces describe it exactly.

2. From forces to wrenches. Shift each generator to B , as in Problem 5. Generator j at contact i becomes a wrench $F_{i,j,W}^B = [\tau_y, f_x, f_z]^T$. Nonnegative combinations of these wrenches give exactly the cone $\mathcal{K}_{\text{total},W}^B$.

3. From a cone to a shape with a size. A cone has no size: if it contains a wrench, it contains every positive multiple of that wrench. Instead, take the *wrench hull* $\mathcal{H} = \text{conv}\{F_{i,j,W}^B\}$, the smallest convex set containing every generator wrench. Roughly, $\mathcal{H}$ holds the wrenches that the fingers can apply together when they share one unit of total normal force. A score then measures how far $\mathcal{H}$ extends from a chosen point.

4. Units. A distance in wrench space mixes Nm with N. Choose a fixed characteristic length ℓ , such as the object's size, and measure in the coordinates $[\tau_y/\ell, f_x, f_z]$, all in newtons.

The parts ask when $\mathcal{H}$ certifies force closure, which measurement of $\mathcal{H}$ is a sensible score with and without gravity, and what is cheap to compute.

Gradescope:

8.1 Select all correct statements.

[1]

- ☒ **The origin strictly inside the full-dimensional hull certifies force closure for the cone generated by these wrenches.**
- ☐ The origin anywhere in the hull, including its boundary, is enough.
- ☐ A nonempty hull certifies force closure.
- ☐ Balancing gravity at the specified pose certifies force closure.

Solution: Only the first statement is correct.

- **Origin strictly inside the full-dimensional hull: true.** A full-dimensional hull has volume in all three wrench coordinates (it is not flat). With the origin strictly inside, some ball of radius $\varepsilon > 0$ about zero lies in the hull. For any nonzero desired wrench F , the vector $\varepsilon F/\|F\|$ lies in that ball and is a convex combination of the generators (nonnegative weights summing to one). Scaling this combination by $\|F\|/\varepsilon$ gives a nonnegative combination producing F , so $F \in \mathcal{K}$, establishing force closure.
- **Origin anywhere, including the boundary: false.** If zero is on the boundary, a supporting plane through it (one with the whole hull on one side) excludes the wrench directions on its other side.

- **Nonempty hull: false.** It says only that at least one wrench is available.
- **Balancing gravity: false.** It tests only the single required wrench $-F_{g,W}^B$, not every direction.

8.2 Which score measures a worst-direction disturbance margin about zero? [1]

- A. The volume of the hull.
- B. The radius of the largest ball centered at the origin contained in the hull.**
- C. The radius of the largest ball, centered anywhere, contained in the hull.

Solution: The ball centered at the origin.

- **Volume: no.** A large volume can still have a narrow direction.
- **Ball centered at the origin: yes.** The fingers resist a disturbance by applying its opposite, and a ball of radius ε about the origin contains every wrench direction up to that magnitude.
- **Ball centered anywhere: no.** A large ball centered elsewhere can leave zero on or outside the hull.

8.3 For a gravity-loaded grasp, where should the ball instead be centered to measure *additional* disturbance margin? [1]

- A. At the origin.
- B. At $F_{g,W}^B$.
- C. At $-F_{g,W}^B$.**
- D. At the centroid of the hull.

Solution: With gravity's wrench $F_{g,W}^B$ and an additional disturbance ΔF , equilibrium requires the contacts to cancel both:

$$F_{\text{contact},W}^B = -F_{g,W}^B - \Delta F.$$

Thus the ball should be centered at $-F_{g,W}^B$, representing the wrench already needed to support the object. A contained ball of radius ε then guarantees resistance to every additional disturbance of norm at most ε . All these norms use the stated torque scaling; otherwise changing the units of torque would change the score. The other choices: the origin ignores the load gravity already places on the grasp; $F_{g,W}^B$ has the wrong sign; and the hull's centroid has no physical meaning here.

8.4 Select all correct statements about practical computation. [1]

- ☒ **Normal/chord alignment is a cheap geometric filter for a parallel-jaw pinch, but does not test reachability or collision.**
- ☐ Every good dexterous-hand grasp must have two opposed, aligned normals.
- ☒ **Given fixed contacts and linearized cones, force balance with bounds can be tested by a linear feasibility program.**
- ☐ Constructing a high-dimensional hull is always as cheap as taking two dot products.

Solution: The first and third statements are correct.

- **Normal/chord alignment is a cheap filter: true.** It is Problem 6's antipodal test (the chord between the contacts lies in each friction cone), and needs only a few dot products at the contacts, making it a useful early filter. It does not use the robot's joints or approach geometry, so it cannot establish reachability or collision avoidance.
- **Every dexterous grasp needs two opposed normals: false.** A dexterous hand may instead combine several contacts whose wrenches balance without any antipodal pair.
- **Force balance is a linear feasibility program: true.** For fixed contacts, collect linearized-cone wrench generators as columns of A . Feasibility takes the form

$$A\lambda = -F_{\text{external},W}^B, \quad \lambda \geq 0, \quad D\lambda \leq b,$$

where λ holds the generators' weights and the last inequalities express linear force limits. This is a linear feasibility program (a linear program with no objective), which solvers answer quickly and reliably.

- **A hull is as cheap as two dot products: false.** Constructing a wrench hull must process many generators and their faces; its cost grows with the number of points and dimension.

Going further: Design a sequence of inexpensive and expensive tests for thousands of candidate grasps. Which failures should be rejected early? How would you compare a large wrench margin with a short, collision-free approach when friction and center of mass are uncertain? How would you choose the torque scale ℓ ?

Solution: Order the tests by cost, so that the cheap ones discard most candidates first:

1. vector checks at the contacts: width within the gripper's opening, the chord inside both friction cones, normals not near vertical;
2. collisions of the gripper at the grasp pose with the object and table;
3. inverse kinematics, to check that the arm can reach the pose;
4. a collision-free approach path;
5. the wrench margin, from a linear program or a hull;
6. finally, simulation of the few best candidates.

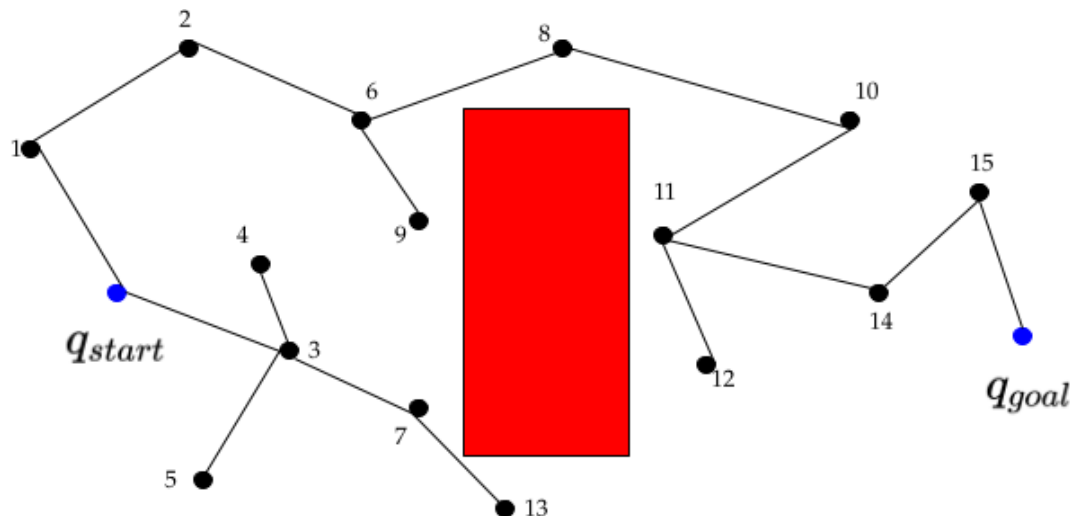
To trade margin against approach, require a minimum margin computed for the worst-case friction and center of mass within their uncertainty. Among the candidates that meet it, prefer the shorter, simpler approach. A large margin at the nominal values matters little if a small error removes it. Choose ℓ as a length of the object, such as its size or the distance from the center of mass to the contacts, so that a torque τ and a force τ/ℓ are comparable disturbances. The score then does not change when the object is described in different units.

1.3 Motion planning

Problem 9 Shortcutting

For this problem, work through Exercise 6.3 in the textbook on improving RRT path quality. Path quality means path length, measured as Euclidean distance, and the extend operator is a straight line.

- 9.1 Below is a tree grown by RRT, in which the path found from q_{start} to q_{goal} is rather jerky. Shortcutting picks two non-consecutive nodes on that path, tries to connect them with a straight line, and keeps the result if it is collision-free and shorter. Select each pair of nodes for which the shortcutting algorithm would result in a shorter path to the goal.



The tree, with the obstacle in red and q_{start} and q_{goal} in blue. Distances are Euclidean in the plane of the figure. (The graph was drawn by hand as an illustrative example.)

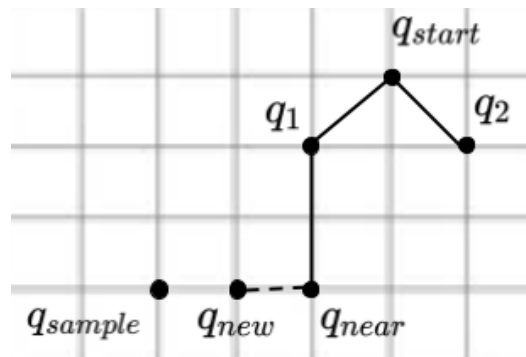
✓ (2, 8)

- ☐ $(q_{start}, 13)$
- ☒ $(1, 6)$
- ☐ $(4, 9)$
- ☒ $(8, 15)$
- ☐ $(12, q_{goal})$

Solution: $(2, 8)$, $(1, 6)$ and $(8, 15)$. The solution path is $q_{start}, 1, 2, 6, 8, 10, 11, 14, 15, q_{goal}$, so a pair is a candidate only if both its nodes lie on it: that rules out $(q_{start}, 13)$, $(4, 9)$ and $(12, q_{goal})$, each of which reaches off the path. The other three are collision-free — $(2, 8)$ and $(8, 15)$ pass above the obstacle — and each straight line is shorter than the detour it replaces, by far the most for $(8, 15)$, which cuts out the zigzag through 10, 11 and 14.

- 9.2 Now zoom in on one iteration of RRT, shown below: q_{sample} is the randomly generated configuration, q_{near} the closest node in the existing tree, and q_{new} the extension from q_{near} toward q_{sample} . When the standard RRT algorithm adds q_{new} to the tree, what node is its parent?

[1]



One extension step. The solid edges are already in the tree; the dashed edge is the extension. Grid cells are unit squares.

- A. q_1
- B. q_2
- C. q_{near}
- D. q_{start}
- E. q_{sample}

Solution: q_{near} — by construction, RRT extends from the nearest node in the tree toward the sample.

- 9.3 If we wanted our tree to encode the shortest path from the starting node, q_{start} , to each node in the tree, what node should be the parent node of q_{new} ? Consider only existing nodes within 3 grid units of q_{new} ; keep the existing tree edges fixed and assume candidate connections are collision-free.

[1]

- A. q_1
- B. q_2
- C. q_{near}
- D. q_{start}
- E. q_{sample}

Solution: q_1 . The right parent is whichever nearby node minimizes its cost-to-come (tree-path length from q_{start}) plus the new edge's length, since that sum is q_{new} 's own cost-to-come. This is the parent-selection step in RRT* (the variant that converges to shortest paths). Its subsequent rewiring step considers changing existing nearby nodes' parents to q_{new} when that reduces their costs. Reading off the grid, cost-to-come is $\sqrt{2} \approx 1.41$ at q_1 and $\sqrt{2} + 2 \approx 3.41$ at q_{near} , so keeping q_{near} as the parent costs 4.41 and going via q_1 costs $\sqrt{2} + \sqrt{5} \approx 3.65$; via q_2 it is 5.02. The direct connection from q_{start} would cost $\sqrt{13} \approx 3.61$, slightly less than the route via q_1 , but q_{start} lies outside the specified 3-unit neighborhood and is therefore not an eligible parent.

2 Computational questions

2.1 Grasping

Problem 10 Normal estimation from a depth image

[6]

Work in `normal_estimation.py`. We use neighboring pixels of a depth image to find nearby points, avoiding a separate nearest-neighbor search. The scene contains a mustard bottle, and the given mask identifies its pixels. The helper `load_camera_data()` returns the depth image, mask, camera intrinsics, and camera pose ${}^W X^C$. Use the calls in the `__main__` block at the end of the file to run the steps below.

- 10.1 Run `show_input()` to see the bottle's depth image, with a colorbar in meters and black outside the mask. Then run `illustrate_window()` to see the estimator's pixel windows outlined in red, zoomed in on the bottle. Close each plotting window to continue. We will estimate a surface normal from each patch of points.

Study `project_depth_to_pC`, which takes rows (v, u, Z) and the camera intrinsics. Then read `depth_points_in_camera_frame`, which takes a depth image and the intrinsics and returns an $(H, W, 3)$ array of camera-frame points. Image arrays are indexed by row and column (v, u) ; u increases rightward and v downward. A depth Z maps to

$${}^C p^P = \begin{bmatrix} (u - c_x)Z/f_x \\ (v - c_y)Z/f_y \\ Z \end{bmatrix}.$$

These points are expressed in the camera frame C , not in world coordinates.

- 10.2 For a window containing points $p_i = {}^C p^{P_i}$, define

[1]

$$\bar{p} = \frac{1}{n} \sum_i p_i, \quad W = \sum_i (p_i - \bar{p})(p_i - \bar{p})^T.$$

W is the 3×3 scatter matrix of the window's points: each eigenvalue measures how spread out the points are along its eigenvector. Suppose the points lie on a plane and are spread out in two directions within it, not bunched along a line or at a single point (a *nondegenerate* patch: only then does it pick out one plane).

Gradescope: For such a patch, which eigenvalue of W has the eigenvector that estimates the normal? Select one.

- A. Smallest eigenvalue.
- B. Middle eigenvalue.
- C. Largest eigenvalue.

Does a matrix whose columns are three unit eigenvectors always represent a proper rotation?

Answer (yes/no): No.

Solution:

- **Which eigenvalue: smallest.** Each eigenvalue of W is the points' spread (sum of squared distances from $\bar{p}$) along its eigenvector, and points on a plane have no spread along the normal.
- **Always a proper rotation: no.** Recall that a proper rotation has orthonormal columns with determinant +1 (a right-handed basis). Eigenvectors of the symmetric W are orthonormal, but each has an arbitrary sign; flipping one gives determinant -1 , a reflection.

10.3 Implement `estimate_normal_by_nearest_pixels`. There is nothing to submit for this part: part 10.4 grades it. Use `normal_patches(pC, mask, uv_step)`, passing through the supplied `uv_step`. It uses `scanning_windows` to visit masked centers on the mask's bounding-box grid in row-major order. With the defaults `UV_STEP=10` and `WINDOW_HALF_WIDTH=3`, the windows are 7×7 before clipping to the image bounds. Each patch contains only masked points with finite coordinates and positive depth. Skip patches with fewer than three points or with second-smallest eigenvalue at most 10^{-12} (in m^2). This rejects patches that do not determine a plane normal. Use `np.linalg.eigh` and orient the normal toward the camera: in C , choose its sign so that $n \cdot \bar{p} \leq 0$, with both n and $\bar{p}$ expressed in C . Construct a right-handed frame N with this normal as $+z$. Put its origin at the patch centroid. Return the list of world poses ${}^W X^N = {}^W X^C {}^C X^N$ in iterator order. Run `test1()` to display the normal frames, camera frame, and camera frustum in Meshcat. It also prints the number of normal frames. We grade each frame's origin and $+z$ axis; rotation about the normal, and hence the choice of tangent axes, is arbitrary.

10.4 Run `test2()`. This evaluates your estimator on a fixed synthetic 3×3 patch without loading the scene or starting Meshcat, so the graded answer does not depend on a renderer. [5]

Gradescope: Enter the printed synthetic frame's translation and $+z$ axis, both expressed in W . Use x, y, z component order and four decimal places.

$${}^W p^N = [\underline{0.1000}, \underline{-0.2000}, \underline{1.3000}]^T \text{ m.}$$

The $+z$ axis of N , expressed in W , is

$$[\underline{0.0000}, \underline{0.0000}, \underline{-1.0000}]^T.$$

Solution: Translation $(0.1, -0.2, 1.3)$ m; normal $(0, 0, -1)$. The synthetic patch is the plane $z_C = 1$, with centroid $(0, 0, 1)$ in C . ${}^W X^C$ translates by $(0.1, -0.2, 0.3)$, so the centroid lands at $(0, 0, 1) + (0.1, -0.2, 0.3) = (0.1, -0.2, 1.3)$ in W . The plane's normal is $\pm(0, 0, 1)$; orienting it toward the camera means $n \cdot \bar{p} \leq 0$, which picks $(0, 0, -1)$. A translation moves points but not directions, so the normal is the same in W .

Problem 11 Sampling-based antipodal grasping

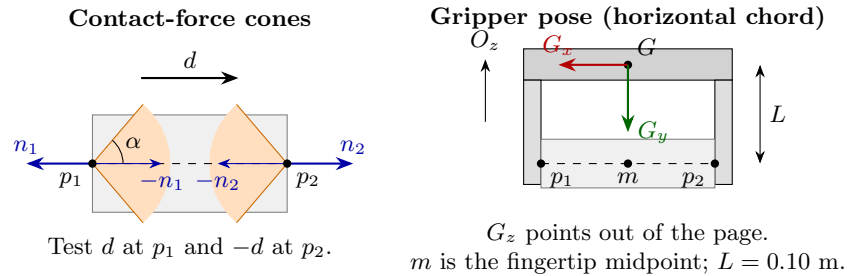
[8]

Work in `sampling_antipodal_grasps.py`. We are given a mesh in object frame O and its true world pose ${}^W X^O$. The robot should pick up the first letter, turn it to match the orientation of the second, and put it down. Start with the provided pair of B's; try your own initials after that works. The letter meshes are generated when the code runs, which needs the letter-mesh packages: in a local venv run `uv pip install "manipulation[letter]"` `scipy` `rtree` (Colab's `setup()` installs them).

11.1 Run `test1()` to inspect the scene. Read the station and controller code. Then study how the tracking diagram is assembled in `build_system_tracking_keyframes`. The controller uses the pseudo-inverse Jacobian as in PS2. It does not enforce joint limits or plan collision-free paths.

11.2 Implement `sample_colinear_points`. First sample the mesh surface using `trimesh.sample.sample_surface`. Obtain outward normals from the sampled face indices. From each point, cast a ray into the object along the negative normal. Use a small inward offset to avoid hitting the starting face. To cast the ray, call `mesh.ray.intersects_location(ray_origins, ray_directions)`, which returns every point where each ray crosses the mesh, the index of the ray it came from, and the index of the face it hit ([API documentation](#)). The arguments are arrays of shape $(n, 3)$, so a single ray is one row. These intersections are unordered: keep the nearest positive-distance hit and its face normal. Return tuples (p_1, p_2, n_1, n_2) , all in O . Pass the supplied random generator as the sampling seed. Read the function docstring for tolerances.

- 11.3 Implement `compute_grasp_from_points` and `get_filtered_grasps`. The first turns one candidate into a gripper pose in O ; the second filters the candidates and calls the first on each survivor. Parts 11.4 and 11.5 grade this part and the last one. Each candidate is (p_1, p_2, n_1, n_2) in object frame O , with unit *outward* normals n_1, n_2 . Here μ is the dimensionless coefficient of static friction assumed by our grasp filter. Use the supplied `mu` argument; its default is `LETTER_MU_STATIC = 1.17`. The inward contact-force cone has half-angle $\alpha = \tan^{-1} \mu$, measured from $-n_i$.



compute_grasp_from_points: construct the pose in O . The gripper's x axis is its closing axis; $+y$ points from its body toward the fingertips. Let width $w = \|p_1 - p_2\|$ and midpoint $m = (p_1 + p_2)/2$. Let k be the upward unit vector along the object frame's positive z axis, expressed in O . Return `None` if $w < 10^{-9}$ m. Also return `None` if the chord is nearly vertical: its unit direction has a horizontal component with norm less than 10^{-6} . Choose $+G_x$ from p_2 toward p_1 , and choose $+G_y$ as close to downward as orthogonality permits. Complete a right-handed frame. Use these axes as the *columns* of ${}^O R^G$ and return a `RigidTransform` with an appropriate translation. Letting $L = \text{FINGER_LENGTH} = 0.10$ m, the pinch point has coordinates $[0, L, 0]^T$ in G . The letters lie flat, so $O_z = W_z$ in this scene.

get_filtered_grasps: filter candidates. The central test is that the chord lies in both friction cones. For $d = (p_2 - p_1)/w$, check

$$(-n_1)^T d \geq \cos \alpha, \quad (-n_2)^T (-d) = n_2^T d \geq \cos \alpha.$$

Also require

$$\text{min_pt_dist} \leq w \leq \text{max_pt_dist}, \quad |n_1^T k| \leq \text{z_axis_thresh}.$$

The last test rejects contacts whose first normal is too close to vertical; it is applied to n_1 only. Reject coincident points before dividing by w , and skip candidates for which `compute_grasp_from_points` returns `None`.

get_filtered_grasps: test for collisions. Transform each surviving pose by ${}^W X^G = {}^W X^O {}^O X^G$, then call the given `check_collision_free`. It checks that all eight corners of a gripper-body box have $z_W > 0$; it does not test finger/mesh, arm, or approach-path clearance. Return a list of accepted world poses in input order (an empty list if none survive).

Run `test2()` to visualize a handful of surviving poses. Would these all work effectively to pick up the object?

- 11.4 Implement `grasp_moment_arm`. Given the gripper body pose X_{WG} and the object's center of mass p_{Wcom} , expressed in world coordinates, return the horizontal distance between the pinch point and the center of mass, in meters. Measure this distance in the world x - y plane. Recall that the pinch point is at $[0, L, 0]^T$ in G ; it is not the gripper body origin.

Why this matters: with two ideal point contacts, the grasp cannot resist torque about the line joining the contacts, which is the G_x axis. We therefore prefer grasps with the pinch point close to the object's center of mass, to reduce the torque caused by gravity.

The given `sample_grasp` selects the candidate with the smallest value. This minimizes the magnitude of the gravitational torque about the pinch point; it is a heuristic for grasp selection, not a guarantee of stability.

Gradescope: Run `test3()`, which uses fixed geometry. Enter the reported lengths to four decimal places, and report whether the grasp passes the table-clearance test.

[2]

Grasp width: 0.0200 m.
 Moment arm: 0.0500 m.
 Table clearance (yes/no): Yes.

Solution: 0.0200 m, 0.0500 m, yes. The pinch point (midway between the fingertips) is at (0,0,0.05) and the center of mass at (0.03,0.04,0.05), so the moment arm, their horizontal distance, is $\sqrt{0.03^2 + 0.04^2} = 0.05$ m. All eight gripper-box corners have $z_W > 0$, so the grasp clears the table.

- 11.5 Implement **plan**. Return the keyframes to approach with open fingers, close while stationary, lift, turn, lower, open while stationary, retreat, and return to the initial pose. Each keyframe is a pair (X_WG, opening): a gripper pose, and the finger opening to command there, which is always one of the constants OPENED or CLOSED defined in the script. Get the poses from the given **compute_prepick_pose** (the approach and retreat) and **lift** (up by LIFT_HEIGHT, and back down by passing -LIFT_HEIGHT), and turn by TURN_ANGLE; the turn is a positive 30° rotation about G_y , which points downward for an overhead grasp, hence a clockwise turn viewed from above. Compose the turn about the *pinch point* so it does not translate during this in-place rotation. Run **test4()**, which should report zero pinch displacement and a -30° world yaw change, then **test5()**.
 Set INITIALS to your own two initials and run **test5()**. Inspect the motion and try another grasp if the object slips or the pick-and-place fails.

Gradescope: Submit a video showing the robot picking up your first initial, turning it to match the orientation of the second, placing it back on the table, and retreating. Both initials should be visible, and the video should show the complete motion.

Play the simulation back in Meshcat in your browser and take a screen recording. Upload it to Gradescope as an mp4 file, much smaller than 500 MB. Gradescope should indicate “Submitted”; the points will be assigned when the staff grades the video.

2.2 Additional problems for 6.4212

Problem 12 Analytic antipodal grasping 6.4212 only

Work in `analytic_antipodal_grasps.py`. Run **test1()** and read `symbolic_derivatives_demo`, which demonstrates Drake’s symbolic Jacobian and Hessian. The supplied `shape(t)` describes a smooth gear boundary $p(t) = {}^O p^{P(t)}$ in a planar object frame.

Here we seek distinct points with a chord normal to the boundary at both ends. This is the normal-aligned special case of an antipodal grasp; with friction, a chord can lie inside both friction cones without being exactly normal. On a nonconvex boundary, stationarity alone does not even guarantee that the chord points into the object at both ends. Define

$$E(t_1, t_2) = \frac{1}{2} \kappa \|p(t_1) - p(t_2)\|^2, \quad \kappa = 2.$$

- 12.1 **Gradescope:** For a normal-aligned antipodal pair, why is $\nabla E = 0$? Select one. [1]
- A. The chord is tangent at both ends.
 - B. The chord is orthogonal to both tangents.**
 - C. The points coincide.

Solution: The chord is orthogonal to both tangents: $E_{t_1} = \kappa(p(t_1) - p(t_2))^T p'(t_1)$ and $E_{t_2} = -\kappa(p(t_1) - p(t_2))^T p'(t_2)$. Each is the dot product of the chord $p(t_1) - p(t_2)$ with a tangent $p'(t_i)$, so both vanish exactly when the chord is normal to the boundary at both ends.

- 12.2 **Gradescope:** Select all reasons that zero gradient alone can fail to certify a usable grasp. [1]
- ✓ **The contact points coincide.**
 - ✓ **A boundary tangent vanishes.**
 - ✓ **The chord points outward at one endpoint.**

✓ **The gripper collides during its approach.**

Solution: All four.

- **Points coincide.** The chord is zero, so it is trivially perpendicular to everything.
- **A tangent vanishes.** If $p'(t) = 0$ the boundary has no defined direction there, so “perpendicular to the tangent” says nothing.
- **Chord points outward.** $\nabla E = 0$ only says the chord is perpendicular to the boundary, not which way it points; on a nonconvex shape it can leave the object at one end, and a finger there would pull rather than push.
- **Gripper collides.** A separate robot feasibility issue: the math knows nothing about the gripper.

12.3 Implement `find_antipodal_pts` using symbolic derivatives and `MathematicalProgram`. This is a feasibility problem, not minimization: [1]

$$\text{find } (t_1, t_2) \quad \text{s.t.} \quad \nabla E = 0, \quad 0 \leq t_i \leq 2\pi - \varepsilon, \quad t_1 - t_2 \geq \varepsilon, \quad \varepsilon = 10^{-3}.$$

Use at most `max_attempts` random initial guesses from the supplied `rng`. A successful solver status is not enough: verify the residual, parameter bounds, nonzero tangents, distinct points, and inward-facing chord at both contacts using `is_antipodal_pair`. This given helper assumes a simple closed boundary; `orientation` is +1 for counterclockwise parameterization and -1 for clockwise (the gear uses -1). Return the accepted parameters and the sorted eigenvalues of the Hessian, using `np.linalg.eigvalsh`. Raise `RuntimeError` if no pair is found. Run `test2()` to inspect a gear grasp.

Gradescope: Is this a convex optimization problem? (yes/no): No.

Solution: No. Recall that equality constraints are convex only when they are linear. The stationarity equations $\nabla E = 0$ involve $p(t)$, built from sines and cosines of t_1, t_2 , so they are nonlinear, with isolated solutions that a local solver finds only from a nearby start: hence the random restarts.

12.4 Run `test3()` to analyze fixed stationary pairs on an ellipse $p(t) = (2 \cos t, \sin t)$, using the same $\kappa = 2$. It prints the Hessian’s eigenvalues at each pair. Use them to classify the pair. [1]

12.4.1 **Gradescope:** Pair $(t_1, t_2) = (0, \pi)$ is a:

- A. minimum **B. maximum** C. saddle D. inconclusive

12.4.2 **Gradescope:** Pair $(t_1, t_2) = (\pi/2, 3\pi/2)$ is a:

- A. minimum B. maximum **C. saddle** D. inconclusive

Solution: Recall the second-derivative test: at a stationary point, Hessian eigenvalues all positive give a minimum, all negative a maximum, and mixed signs a saddle.

- $(0, \pi)$, **the ends of the long (major) axis: maximum.** Eigenvalues $(-16, -12)$, both negative.
- $(\pi/2, 3\pi/2)$, **the ends of the short (minor) axis: saddle.** Eigenvalues $(-4, 12)$, one of each sign. In particular, it is not a local minimum when both contact parameters vary independently.

12.5 **Gradescope:** At a stationary pair, select all valid conclusions. [1]

- ✓ **A positive-definite Hessian gives a strict local minimum of E .**
☐ A local maximum of E means the boundary is concave at both contacts.
 ✓ **A zero Hessian eigenvalue makes this second-derivative test inconclusive.**
☐ Minimizing this energy alone certifies a stable grasp by the real robot.

Solution: First and third.

- **Positive-definite Hessian gives a strict local minimum: true.** This is the second-derivative test.

- *Local maximum means concave at both contacts: false.* E 's curvature is about the distance between two points as both move, not about whether the boundary bulges in or out at either one; the ellipse's major-axis pair is a maximum on a convex boundary.
- *Zero eigenvalue makes the test inconclusive: true.* E is flat to second order in that direction, so higher derivatives decide.
- *Minimizing E certifies a stable grasp: false.* E models neither friction, nor finite fingertips, nor how a real gripper can move, so it is not a complete model of grasp stability.

Run `test4()` to look for examples of different Hessian signatures on the gear. Discuss what this energy model leaves out about friction, finite fingertips, and the motions permitted by a real gripper.

3 What you should understand about the code you just produced

You should be able to answer these questions without running the code, using a short explanation, sketch, or calculation. Written tests may ask you to apply these ideas to a different geometry or situation.

- How does a depth discontinuity affect PCA? What changes when a window contains a narrow strip of points rather than a surface patch?
- Two implementations return opposite normal vectors for the same patch. Can both be valid PCA results? How do we choose the sign? Can two valid normal frames still have different tangent axes?
- The camera is translated without changing its orientation. How does transforming a patch centroid from camera coordinates into world coordinates differ from transforming its normal? What goes wrong if you transform both as points?
- What is the role of comparisons with `min_pt_dist` and `max_pt_dist` in the grasp filtering?
- Which of our grasp filters express physical constraints, and which are heuristics? What failures can survive all of them?
- Explain the signs in $(-n_1)^T d \geq \cos \alpha$. Write the corresponding test at the second contact. Why must the normals and d be unit vectors?
- What if we positioned the gripper's body origin at the midpoint between the two desired contact points? Would the fingertips reach those points? Use a sketch to explain where the body origin should be instead.
- What if we changed the gripper's orientation while keeping its body origin fixed? Would the midpoint between the fingertips stay fixed? Sketch what happens, and explain how the body's position must change to keep that midpoint fixed.
- A grasp passes our `check_collision_free` function, but the robot hits the object or table during execution. How is that possible? Describe what the function actually checks and two additional checks needed for the complete motion.
- Why can a small gravity moment arm be useful without certifying force closure?
- A grasp can support the object against gravity in its initial pose but does not have force closure. What could go wrong when the robot rotates the object? What could go wrong if it follows the same path faster? What must a planner check beyond collisions?
- **6.4212:** The solver reports success and $\nabla E = 0$. Give two reasons the returned pair might still be invalid. If the pair is valid, does the Hessian's classification tell you whether the real grasp is stable?